Published by Institute of Physics Publishing for SISSA

Received: June 12, 2008
Accepted: July 30, 2008
Published: August 7, 2008

Exact half-BPS flux solutions in M-theory I local solutions*

Eric D'Hoker, John Estes, Michael Gutperle and Darya Krym

Department of Physics and Astronomy, University of California, Los Angeles, 405, Hilgard Avenue, Los Angeles, CA 90095, U.S.A.
E-mail: dhoker@physics.ucla.edu, estes@physics.ucla.edu, gutperle@physics.ucla.edu, daryakrym@gmail.com

Abstract: The complete eleven-dimensional supergravity solutions with 16 supersymmetries on manifolds of the form $AdS_3 \times S^3 \times S^3 \times \Sigma$, with isometry $\mathrm{SO}(2,2) \times \mathrm{SO}(4) \times \mathrm{SO}(4)$, and with either $AdS_4 \times S^7$ or $AdS_7 \times S^4$ boundary behavior, are obtained in exact form. The two-dimensional parameter space Σ is a Riemann surface with boundary, over which the product space $AdS_3 \times S^3 \times S^3$ is warped. By mapping the reduced BPS equations to an integrable system of the sine-Gordon/Liouville type, and then mapping this integrable system onto a linear equation, the general local solutions are constructed explicitly in terms of one harmonic function on Σ, and an integral transform of two further harmonic functions on Σ. The solutions to the BPS equations are shown to automatically solve the Bianchi identities and field equations for the 4-form field, as well as Einstein's equations. The solutions we obtain have non-vanishing 4-form field strength on each of the three factors of $AdS_3 \times S^3 \times S^3$, and include fully back-reacted M2-branes in $AdS_7 \times S^4$ and M5-branes in $AdS_4 \times S^7$. No interpolating solutions exist with mixed $AdS_4 \times S^7$ and $AdS_7 \times S^4$ boundary behavior. Global regularity of these local solutions, as well as the existence of further solutions with neither $AdS_4 \times S^7$ nor $AdS_7 \times S^4$ boundary behavior will be studied elsewhere.

Keywords: AdS-CFT Correspondence, M-Theory.

*This work was supported in part by National Science Foundation (NSF) grants PHY-04-56200 and PHY-07-57702.

JHEP08(2008)028

Contents

1. Introduction

The AdS/CFT correspondence [1 – 3] (for reviews, see [4, 5]) maps local and non-local gauge invariant operators on the CFT side onto solutions to supergravity on the AdS side. In the 't Hooft limit of large gauge group, and for large 't Hooft coupling, the CFT operators map to solutions of classical supergravity. As a result, the knowledge of certain classical supergravity solutions can benefit our understanding of the dynamics on the CFT side.

Substantial progress has been made over the past few years in spelling out this correspondence for the special case where 16 supersymmetries are preserved by both the operators on the CFT side, and the supergravity solutions on the AdS side. In 10-dimensional Type IIB supergravity, infinite families of solutions which are invariant under 16 supersymmetries have been obtained in exact form. Each one of these solutions is dual to a particular gauge invariant operator in 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory.

A first family consists of exact solutions [6] dual to local gauge invariant half-BPS operators [7]. A second family consists of exact solutions [8, 9] which generalize the Janus solution with no supersymmetry of [10], and the Janus solution with 4 supersymmetries of [11, 12]. The solutions in this second family are dual to half-BPS planar interface operators [13], generalized to include varying coupling constant in [14 – 16], and further generalized to include also varying instanton θ-angle in [17, 18]. A third family consists of exact solutions [19] dual to half-BPS Wilson loops [20]. (Earlier work [21 – 23] includes a derivation of the reduced BPS equations, and a study of boundary conditions.)

The knowledge of these exact solutions should provide a powerful starting point for the study of the spectrum of small fluctuations and of correlation functions, either exactly or numerically, and thus for the spectrum and correlation functions of the dual CFT.

In the present paper we shall extend the construction of exact solutions with 16 supersymmetries to the case of 11-dimensional supergravity, or M-theory. The AdS/CFT correspondence for M-theory actually produces two dualities. The first duality maps $AdS_4 \times S^7$ to a 3-dimensional CFT with 32 supersymmetries, whose canonical bosonic fields are scalars. The second duality maps $AdS_7 \times S^4$ to a 6-dimensional CFT with 32 supersymmetries, whose canonical bosonic fields are 2 forms with self-dual field strength. Our present understanding of both of these CFTs is much weaker than our understanding of 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory.

The difficulty of obtaining classical supergravity solutions with 16 supersymmetries is, however, roughly comparable to the difficulty of obtaining the corresponding solutions in Type IIB supergravity. One may hope that by improving our knowledge of exact solutions in M-theory we will be able to deepen our understanding of the dynamics of the corresponding more elusive CFTs. Currently, progress is being made towards constructing an effective field theory for multiple M2 branes [24 – 26]. It will be interesting to see precisely how the half-BPS supergravity solutions we find in the present paper are dual to half-BPS planar interface operators in the M-brane theory.

Specifically, we shall construct, in exact form, the complete local solution invariant under 16 supersymmetries in 11-dimensional supergravity with the geometry $AdS_3 \times S^3 \times S^3 \times \Sigma$ invariant under $\mathrm{SO}(2,2) \times \mathrm{SO}(4) \times \mathrm{SO}(4)$, and boundary asymptotics of either $AdS_4 \times S^7$ or $AdS_7 \times S^4$. Solutions with this boundary behavior are the most immediately relevant in the context of the AdS/CFT correspondence. Their construction is also technically simpler than that of solutions with more general boundary conditions. The manifold Σ is a 2-dimensional parameter space over which the space $AdS_3 \times S^3 \times S^3$ is, generally, warped. The special case of solutions with space-time $AdS_3 \times S^3 \times S^3 \times E_2$, where E_2 is flat Euclidean, and the product is not warped, was analyzed in [27] (see also [28, 29]). Other types of solutions on various space-times with an AdS_3 factor, and various degrees of supersymmetry, have been constructed in [30 – 32].

The Ansatz $AdS_3 \times S^3 \times S^3 \times \Sigma$ encompasses as special cases both the solutions $AdS_4 \times S^7$ and $AdS_7 \times S^4$ with the maximum of 32 supersymmetries. For these special cases, the supergravity 4-form field strength F vanishes on two out of the three factors $AdS_3 \times S^3 \times S^3$, and is non-vanishing and constant on the third factor.

The solutions we shall obtain here will have a non-vanishing 4-form field strength F on

JHEP08(2008)028

all three factors of $AdS_3 \times S^3 \times S^3$, thus allowing for non-zero M2- and M5-brane charges. Such supergravity field configurations include fully back-reacted solutions of M2-branes in $AdS_7 \times S^4$ and M5-branes in $AdS_4 \times S^7$. In principle, such solutions could also be viewed as Maldacena limits of brane configurations with 8 supersymmetries. Indeed, the intersection of a stack of coincident M2-branes with a stack of coincident M5-branes over a 2-dimensonal string worldsheet is 1/4-BPS, and thus leaves 8 supersymmetries. The Maldacena limit, either nearing the M2 horizon or nearing the M5 horizon, will produce 8 additional (conformal) supersymmetries, bringing the total to 16 supersymmetries [27 – 29]. The problem is that the corresponding fully localized intersecting M2/M5-brane solution is not (yet) known.

The search for such solutions invariant under 16 supersymmetries was initiated by Yamaguchi [21], and by Lunin in [33] where the BPS equations were reduced to an $AdS_3 \times S^3 \times S^3 \times \Sigma$ Ansatz, a harmonic function on Σ was identified, a semi-quantitative investigation into the boundary conditions was carried out, and arguments for the existence of solutions were presented, based on a perturbative expansion. Obtaining complete solutions there, however, would still require solving non-linear partial differential equations, which was not done in [21, 33]. The main novelty of the present paper lies in the following results:

(1) The reduced BPS equations are completely solved, in exact form;

(2) It is shown that any solution to the BPS equations automatically solves the M-theory Bianchi and field equations.

(3) Two further harmonic functions are identified in the construction of the local solution.

Examination of the reduced BPS equations reveals (a fact also observed in [21, 33]) that the space of solutions is naturally foliated by three constant real parameters c_i, with $i = 1, 2, 3$, which are subject to the relation, $c_1 + c_2 + c_3 = 0$, but are otherwise free. The absolute values $|c_i|$ correspond to the inverse radii of each of the three factors in $AdS_3 \times S^3 \times S^3$. To make this correspondence precise, it will be useful to label the factors as follows, $AdS_3 \times S^3_2 \times S^3_3$, where the AdS_3 factor corresponds to label 1, but this label will not be exhibited. Thus, $|c_1|$ is the inverse radius of AdS_3, while $|c_i|$ is the radius of S^3_i with $i = 2, 3$. A simultaneous rescaling of all three c_i corresponds to an overall rescaling of the solution size. The space of the constant parameters c_i modulo their overall rescaling consists of just a single free real parameter c which plays the role of an *aspect ratio* of the solution. The dependence on c of the warping of $AdS_3 \times S^3_2 \times S^3_3$ over Σ is highly non-trivial.

In the limit where one of the parameters c_i tends to zero, the radius of the factor space with label i in $AdS_3 \times S^3_2 \times S^3_3$ tends to infinity with the effect of decompactifying that space. Due to the relation $c_1 + c_2 + c_3 = 0$, it is not possible for two factors to decompactify simultaneously, while the third factor is left at finite radius. But it is possible for all three c_i to tend to zero, in which case the entire solution decompactifies. A schematic representation of the 2-dimensional parameter space generated by c_1, c_2, c_3 is depicted in figure 1 below.

The special solutions $AdS_4 \times S^7$ and $AdS_7 \times S^4$, with the maximal number of 32 supersymmetries, correspond to two of the parameters c_i being coincident ($c_2 = c_3$ for the first case, and $c_1 = c_2$ or $c_1 = c_3$ for the second case). The constants c_i are determined

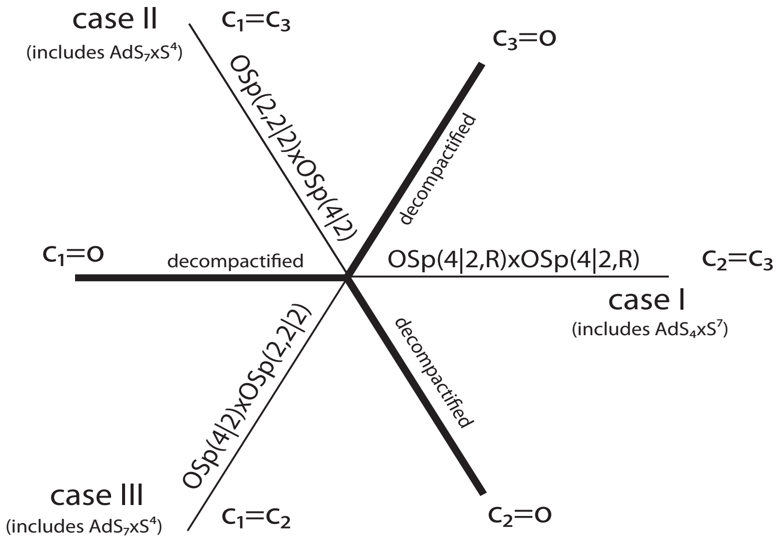

Figure 1: The space of parameters c_1, c_2, c_3. Assignments differing only by an overall sign have been identified in this representation.

case	relation	includes	supergroup of solution	maximal supergroup
I	$c_2 = c_3$	$AdS_4 \times S^7$	$\mathrm{OSp}(4\|2, \mathbf{R}) \times \mathrm{OSp}(4\|2, \mathbf{R})$	$\mathrm{OSp}(8\|4, \mathbf{R})$
II	$c_3 = c_1$	$AdS_7 \times S^4$	$\mathrm{OSp}(2, 2\|2) \times \mathrm{OSp}(4\|2)$	$\mathrm{OSp}(2, 6\|4)$
III	$c_1 = c_2$	$AdS_7 \times S^4$	$\mathrm{OSp}(4\|2) \times \mathrm{OSp}(2, 2\|2)$	$\mathrm{OSp}(2, 6\|4)$

Table 1: Cases with either $AdS_4 \times S^7$ or $AdS_7 \times S^4$ boundary behavior.

completely by the boundary behavior of the solution. It follows that any solution with $AdS_4 \times S^7$ boundary behavior on all or on part of its boundary must have $c_2 = c_3$. Any solution with $AdS_7 \times S^4$ boundary behavior must have either $c_1 = c_2$ or $c_1 = c_3$. It also follows that solutions for which no two c_i coincide with one another have neither $AdS_4 \times S^7$ nor $AdS_7 \times S^4$ boundary behavior. Finally, it also follows that no solution can interpolate between $AdS_4 \times S^7$ boundary behavior and $AdS_7 \times S^4$ boundary behavior. (Note that this result does not assume global regularity of the solution.)

The space of solutions we obtain may also be considered from the point of view of the supergroups under which the solutions are invariant. The $SO(2, 2) \times SO(4) \times SO(4)$ isometry group of our Ansatz $AdS_3 \times S_2^3 \times S_3^3 \times \Sigma$ is the maximal bosonic subgroup of the supergroup of the solutions. The cases $AdS_4 \times S^7$ and $AdS_7 \times S^4$ with maximal supersymmetry are invariant under the supergroups $\mathrm{OSp}(8|4, \mathbf{R})$ and $\mathrm{OSp}(2, 6|4)$ respectively. General arguments [34] show that the supergroup under which any solution is invariant must be a subgroup of either $\mathrm{OSp}(8|4, \mathbf{R})$ or $\mathrm{OSp}(2, 6|4)$. The possible supergroups with 16 supersymmetries available for the geometry $AdS_3 \times S_2^3 \times S_3^3 \times \Sigma$ are listed in table 1 below.

In table 1, only the cases where two of the c_i coincide are listed, because these are the only cases that allow for either $AdS_4 \times S^7$ or $AdS_7 \times S^4$ asymptotic boundary behavior. Their designations as cases I, II, or III will be used throughout. In figure 1, the cases I, II, and III are indicated with thin lines. They are of most interest to AdS/CFT and it is for these cases that the complete exact solutions will be obtained in this paper.

The regions in figure 1 where no two c_i coincide with one another correspond to solutions whose boundary behavior is neither that of $AdS_4 \times S^7$ nor that of $AdS_7 \times S^4$. These solution spaces are foliated by the aspect ratio parameter c. As a function of c, the supergroup which leaves the solution invariant will change. For example, across the decompactification lines $c_i = 0$, the boundary conditions change, and so do the corresponding supergroups, by passing through a common Wigner-Inonu. There may also be, however, a more intricate dependence of the supergroups on c, since it is possible to deform OSp(4|2) into the exceptional supergroup $D(2,1,a)$. Thus, in the region where no two c_i coincide, the supergroup of the solution may actually be a non-compact real form of $D(2,1,a) \times D(2,1,b)$ with a, b depending on the aspect ratio c. This possibility would be consistent with the M-brane analysis of [27, 28].

A final result concerns the integrable system onto which the reduced BPS equations are mapped, and from which the exact solutions will be constructed. Its field is a single real scalar function ϑ on Σ. The field equation is given by,

$$4\partial_{\bar{w}}\partial_w \vartheta - \partial_{\bar{w}} \left(ie^{2i\vartheta} \partial_w \ln h \right) + \partial_w \left(ie^{-2i\vartheta} \partial_{\bar{w}} \ln h \right) = 0 \tag{1.1}$$

where h is an arbitrary harmonic function on Σ. This integrable system is of the sine-Gordon/Liouville type, with translation invariance in the direction perpendicular to the coordinate h, but broken translation invariance along the direction h. This field theory is akin to the Liouville theory with broken translation invariance of [35] and shares with it the remarkable property that its full solution may be obtained in explicit form. A similar integrable system was encountered in the solution of Type IIB theory in [8, 9]. Yet, the two systems are different from one another.

The remainder of this paper is organized as follows.

In section 2, we briefly review 11-dimensional supergravity, spell out the $AdS_3 \times S_2^3 \times S_3^3 \times \Sigma$ Ansatz for the supergravity fields and supersymmetry spinor, and produce the reduced BPS equations, which are the starting point of our exact solution.

In section 3, we solve for the metric factors of the $AdS_3 \times S_2^3 \times S_3^3$ spaces, express these metric factors in terms of bilinears in the supersymmetry spinors, and reproduce the maximally supersymmetric solutions $AdS_4 \times S^7$ and $AdS_7 \times S^4$.

In section 4, we derive from the reduced BPS equations, for all values of the constants c_i, a holomorphic 1-form κ on Σ.

In section 5, this holomorphic 1-form κ is used to produce the complete solution to the BPS equations for case III, namely when $c_1 = c_2$. This is achieved by mapping the reduced BPS equations to the integrable system (1.1), and then mapping this system onto a linear equation. The metric factors of the solutions are computed explicitly in terms of the solutions to this linear equation. In section 6, case II is solved by showing that it is related to case III by a simple discrete symmetry of the reduced BPS equations. In section 7, case I is solved by methods identical to the ones used to solve for case III.

In section 8, the remaining linear differential equation is solved exactly in terms of an integral transform of two holomorphic functions on Σ. It is shown that, alternatively, the solution may be obtained by projection from a single 3-dimensional harmonic function.

In section 9, the flux fields are computed as well. It is shown that the Bianchi identities, the field equations for the 4-form field strength, and Einstein's equations all hold automatically for any solution to the BPS equations.

In appendix A, we give a Clifford algebra representation adapted to the $AdS_3 \times S^3 \times S^3 \times \Sigma$ geometry. In appendix B, we summarize the geometry of Killing spinors for odd-dimensional spheres and odd-dimensional Minkowski AdS space-times.

2. M-theory ansatz and reduced BPS equations

In this section, we shall construct the most general $SO(2,2) \times SO(4) \times SO(4)$ invariant Ansatz for the bosonic fields of eleven-dimensional supergravity on the manifold[1] $AdS_3 \times S_2^3 \times S_3^3$ warped over a two-dimensional parameter space Σ. Since Σ inherits an orientation and a metric from supergravity, Σ is automatically a Riemann surface, endowed with a complex structure. The BPS equations will be reduced to this Ansatz and simplified through the use of a Killing spinor basis for the spinors on $AdS_3 \times S_2^3 \times S_3^3$.

2.1 Eleven-dimensional supergravity

Eleven dimensional supergravity [36] contains two bosonic fields, the metric g_{MN}, and a four form field strength F_{PQRS}, with $M, N, P, Q, R, S = 0, 1, \ldots, 9, \natural \equiv 10$. It will be convenient to recast the tensor F_{PQRS} in terms of a 4-form $F = F_{PQRS} dx^P \wedge dx^Q \wedge dx^R \wedge dx^S/24$, which is given in terms of a 3-form potential C by $F = dC$. Supergravity also contains a fermion field, the gravitino Ψ_M, but here we shall be interested in purely bosonic solutions, and thus set $\Psi_M = 0$ throughout. The action is then given by

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right) - \frac{1}{12\kappa_{11}^2} \int C \wedge F \wedge F \tag{2.1}$$

The field equation for the metric g_{MN} is,

$$R_{MN} - \frac{1}{12} F_{MPQR} F_N{}^{PQR} + \frac{1}{144} g_{MN} F_{PQRS} F^{PQRS} = 0 \tag{2.2}$$

The Bianchi identity, and field equation for the 4-form F are respectively,

$$dF = 0 \qquad\qquad d * F + \frac{1}{2} F \wedge F = 0 \tag{2.3}$$

Supersymmetry of a purely bosonic field configuration under a transformation with supersymmetry parameter ε requires that the supersymmetry variation of Ψ_M with respect to ε must vanish for vanishing Ψ_M. This requirement yields the BPS equations,

$$\nabla_M \varepsilon + \frac{1}{288} \left(\Gamma_M{}^{NPQR} - 8 \delta_M{}^N \Gamma^{PQR} \right) F_{NPQR} \, \varepsilon = 0 \tag{2.4}$$

[1] Throughout, we shall use the notation in which the AdS_3 space corresponds to the label $a = 1$ (which will not be exhibited on AdS_3), and the spheres S_a^3 correspond to the labels $a = 2, 3$.

Here, ε is an eleven dimensional Majorana spinor, and ∇_M is the covariant derivative with respect to the Levi-Civita connection for g_{MN}. It will be convenient to use the identity

$$\Gamma^{MNPQR} - 8\delta^{M[N}\Gamma^{PQR]} = -\frac{1}{2}\Gamma^M\Gamma^{NPQR} + \frac{3}{2}\Gamma^{NPQR}\Gamma^M \tag{2.5}$$

to recast the BPS equation in the form,[2]

$$\nabla_M\varepsilon + \frac{1}{24^2}\Big(- \Gamma_M(\Gamma \cdot F) + 3(\Gamma \cdot F)\Gamma_M\Big)\varepsilon = 0 \tag{2.6}$$

The advantage of this form of the BPS equations is that only the combination $\Gamma \cdot F$ appears.

2.2 Invariant ansatz for supergravity fields

The $SO(2,2)\times SO(4)\times SO(4)$ invariant Ansatz for the supergravity fields on $AdS_3 \times S^3_2 \times S^3_3$ warped over Σ are as follows. The Ansatz for the metric is given by,

$$ds^2 = f_1^2\, ds^2_{AdS_3} + f_2^2\, ds^2_{S^3_2} + f_3^2\, ds^2_{S^3_3} + ds^2_\Sigma \tag{2.7}$$

Here, $ds^2_{AdS_3}$, $ds^2_{S^3_2}$, and $ds^2_{S^3_3}$, are the unit radius metrics on the corresponding spaces, respectively invariant under $SO(2,2)$, $SO(4)$ and $SO(4)$. The metric ds^2_Σ is an unspecified Riemannian metric on Σ, and the *metric factors* f_1, f_2, and f_3 are unspecified real (but not necessarily positive) functions on Σ, the expressions for all of which will be determined by imposing the BPS equations. It will be convenient to cast the supergravity metric (and other fields) in terms of an eleven-dimensional Lorentz frame $e^A \equiv dx^M\, e_M{}^A$, with $A = 0, 1, \ldots, 9, \natural$, chosen as follows,

$$\begin{aligned}
e^{i_1} &= f_1\hat{e}^{i_1} & i_1 &= 0,1,2 \\
e^{i_2} &= f_2\hat{e}^{i_2} & i_2 &= 3,4,5 \\
e^{i_3} &= f_3\hat{e}^{i_3} & i_3 &= 6,7,8 \\
e^{a} & & a &= 9,10
\end{aligned} \tag{2.8}$$

The frames $\hat{e}^{i_1}, \hat{e}^{i_2}$, and $\hat{e}^{i_3}$ correspond to the unit radius metrics on AdS_3, S^3_2, and S^3_3 respectively. The Ansatz for the 3-form potential C is conveniently expressed as follows,

$$C = b_1\hat{e}^{012} + b_2\hat{e}^{345} + b_3\hat{e}^{678} \tag{2.9}$$

where b_1, b_2, and b_3 are real functions on Σ. The corresponding field strength takes the form,

$$F = g_{1a}e^{012a} + g_{2a}e^{345a} + g_{3a}e^{678a} \tag{2.10}$$

with

$$g_{ia} = -\frac{1}{f_i^3}D_a b_i \qquad\qquad i = 1,2,3 \tag{2.11}$$

Throughout, the covariant derivative on Σ acting on scalar functions on Σ will be denoted by $D_a \equiv e_a{}^M\partial_M$ where $e_A{}^M$ is the inverse of the Lorentz frame $e_M{}^A$.

[2]Throughout, we shall use the notation $\Gamma \cdot T = \Gamma^{M_1\cdots M_p}T_{M_1\cdots M_p}$ for the contraction of an antisymmetric tensor field of rank p with the Γ-matrix of the same rank.

JHEP08(2008)028

2.3 Invariant ansatz for the supersymmetry parameters

The supersymmetry parameter ε must be compatible with the $\mathrm{SO}(2,2) \times \mathrm{SO}(4) \times \mathrm{SO}(4)$ symmetry of the problem, and globally well-defined on the symmetric spaces AdS_3, S^3_2, and S^3_3. Killing spinors on AdS_3, S^3_2, and S^3_3 provide a convenient basis for the invariant and globally well-defined spinors ε on these symmetric spaces.[3] The factors AdS_3, S^3_2, and S^3_3 each carry two-dimensional irreducible spinors. It will be convenient to organize the invariant spinors on $AdS_3 \times S^3_2 \times S^3_3$ directly in terms of 8-dimensional spinors χ. The Killing spinor equations on $AdS_3 \times S^3_2 \times S^3_3$ take the following form,

$$\begin{aligned}
0 &= \left(\hat{\nabla}_{i_1} - \frac{\eta_1}{2}(\gamma_{i_1} \otimes I_2 \otimes I_2)\right)\chi^{\eta_1,\eta_2,\eta_3} \\
0 &= \left(\hat{\nabla}_{i_2} - i\frac{\eta_2}{2}(I_2 \otimes \gamma_{i_2} \otimes I_2)\right)\chi^{\eta_1,\eta_2,\eta_3} \\
0 &= \left(\hat{\nabla}_{i_3} - i\frac{\eta_3}{2}(I_2 \otimes I_2 \otimes \gamma_{i_3})\right)\chi^{\eta_1,\eta_2,\eta_3}
\end{aligned} \tag{2.12}$$

The covariant derivatives $\hat{\nabla}_{i_1}, \hat{\nabla}_{i_2}$, and $\hat{\nabla}_{i_3}$ are with respect to the unit radius metrics of the corresponding spaces. The parameters η_1, η_2 and η_3 are restricted to take the values ± 1 as a result of requiring that the integrability conditions be satisfied. The equations (2.12) then have solutions of maximal rank, and a basis of Killing spinors is provided by the 8 linearly independent solutions corresponding to $\eta_1 = \pm 1, \eta_2 = \pm 1$, and $\eta_3 = \pm 1$.

The supersymmetry parameter ε may be decomposed in the basis of Killing spinors $\chi^{\eta_1,\eta_2,\eta_3}$, and we shall denote the corresponding coefficients by $\zeta_{\eta_1,\eta_2,\eta_3}$. For a fixed assignment of η_1, η_2, and η_3, the object $\zeta_{\eta_1,\eta_2,\eta_3}$ is a four component spinor, and we have

$$\varepsilon = \sum_{\eta_1,\eta_2,\eta_3} \chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} \tag{2.13}$$

Without loss of generality, we may impose a reality condition on the basis of Killing spinors $\chi^{\eta_1,\eta_2,\eta_3}$, since the coefficients $\zeta_{\eta_1,\eta_2,\eta_3}$ are, in general, allowed to be complex spinors. The proper reality condition is

$$(\chi^{\eta_1,\eta_2,\eta_3})^* = (I_2 \otimes \sigma^2 \otimes \sigma^2)\chi^{\eta_1,\eta_2,\eta_3} \tag{2.14}$$

This reality condition on the Killing spinors $\chi^{\eta_1,\eta_2,\eta_3}$, together with the Majorana condition $\varepsilon^* = B\varepsilon$ on the full 32-component spinor ε implies a corresponding reality condition on the coefficients $\zeta_{\eta_1,\eta_2,\eta_3}$, which is found to be,

$$(\zeta_{\eta_1,\eta_2,\eta_3})^* = (\sigma^3 \otimes \sigma^1)\zeta_{\eta_1,\eta_2,\eta_3} \tag{2.15}$$

Finally, we may use the Killing spinor equations (2.12) to express the eleven-dimensional covariant derivatives along the symmetric spaces in terms of an algebraic action by the

[3] Conventions for Γ matrices are in appendix A, and basic results on Killing spinors for odd dimensional symmetric spaces are collected in appendix B.

eleven-dimensional Γ-matrices as follows,

$$
\begin{aligned}
\nabla_{i_1}\epsilon &= \sum_{\eta_1,\eta_2,\eta_3} \left(+ \frac{\eta_1}{2f_1}\Gamma_{i_1}\Gamma^{012} + \frac{1}{2}\omega_{i_1 j_1 a}\Gamma^{j_1 a}\right)\chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} \\
\nabla_{i_2}\epsilon &= \sum_{\eta_1,\eta_2,\eta_3} \left(- \frac{\eta_2}{2f_2}\Gamma_{i_2}\Gamma^{345} + \frac{1}{2}\omega_{i_2 j_2 a}\Gamma^{j_2 a}\right)\chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} \\
\nabla_{i_3}\epsilon &= \sum_{\eta_1,\eta_2,\eta_3} \left(- \frac{\eta_3}{2f_3}\Gamma_{i_3}\Gamma^{567} + \frac{1}{2}\omega_{i_3 j_3 a}\Gamma^{j_3 a}\right)\chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3}
\end{aligned}
\tag{2.16}
$$

These equations will guarantee that the 9 space-time vector components of the BPS equations along the directions of $AdS_3 \times S_2^3 \times S_3^3$ will result in algebraic conditions on $\zeta_{\eta_1,\eta_2,\eta_3}$.

2.4 The reduced BPS equations

To reduce the BPS equations to the $SO(2,2) \times SO(4) \times SO(4)$ invariant Ansatz, we need the components of the 11-dimensional Lorentz connection $\omega^A{}_B$. They are given as follows,

$$
\begin{aligned}
\omega^{i_1}{}_{j_1} &= \hat{\omega}^{i_1}{}_{j_1} & \omega^{i_1}{}_a &= e^{i_1}\frac{D_a f_1}{f_1} \\
\omega^{i_2}{}_{j_2} &= \hat{\omega}^{i_2}{}_{j_2} & \omega^{i_2}{}_a &= e^{i_2}\frac{D_a f_2}{f_2} \\
\omega^{i_3}{}_{j_3} &= \hat{\omega}^{i_3}{}_{j_3} & \omega^{i_3}{}_a &= e^{i_3}\frac{D_a f_3}{f_3}
\end{aligned}
\tag{2.17}
$$

together with the connection on Σ, given by $\omega^a{}_b = \hat{\omega}\epsilon^a{}_b$ for which we use the convention that $\epsilon^{9\natural} = \epsilon^9{}_\natural = +1$. All other components vanish. The connections with hats refer to the connections on the unit radius manifolds AdS_3, S_2^3, and S_3^3 respectively.

The Killing spinor equations on the basis spinors $\chi^{\eta_1,\eta_2,\eta_3}$ allow us to recast the action of the covariant derivatives ∇_{i_1}, ∇_{i_2}, and ∇_{i_3} on the supersymmetry parameter ε in terms of a purely algebraic linear action on ε. To simplify the action of the covariant derivatives ∇_a on ε along Σ, we make use of

$$
\nabla_a \chi^{\eta_1,\eta_2,\eta_3} \otimes \zeta_{\eta_1,\eta_2,\eta_3} = \chi^{\eta_1,\eta_2,\eta_3} \otimes \left(D_a \zeta_{\eta_1,\eta_2,\eta_3} + \frac{i}{2}\hat{\omega}_a (1 \otimes \sigma^3)\zeta_{\eta_1,\eta_2,\eta_3}\right) \tag{2.18}
$$

To simplify the terms in the BPS equation that involve F, we use the following relation,

$$
\frac{1}{24}\Gamma \cdot F = g_{1a}\Gamma^{012a} + g_{2a}\Gamma^{345a} + g_{3a}\Gamma^{678a} \tag{2.19}
$$

where the multiple Γ-matrices take the form, (see appendix A for Γ-matrix conventions),

$$
\begin{aligned}
\Gamma^{0123} &= +I_8 \otimes \sigma^1 \otimes \sigma^3\sigma^a \\
\Gamma^{345a} &= -iI_8 \otimes \sigma^2 \otimes \sigma^3\sigma^a \\
\Gamma^{678a} &= -iI_8 \otimes \sigma^3 \otimes \sigma^3\sigma^a
\end{aligned}
\tag{2.20}
$$

Here, I_8 is the identity matrix in the 8-component spinor space generated by the Killing spinors $\chi^{\eta_1,\eta_2,\eta_3}$, while the last two factors in the tensor product act on the 4-component spinors $\zeta_{\eta_1,\eta_2,\eta_3}$. The resulting reduced BPS equations take the following form.

On the AdS_3 space, we have,

$$0 = \frac{\eta_1}{2f_1}(\sigma^1 \otimes \sigma^3)\zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_1}{f_1}(1 \otimes \sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} \\ -\frac{i}{12}g_{2a}(\sigma^2 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} - \frac{i}{12}g_{3a}(\sigma^3 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} - \frac{1}{6}g_{1a}(\sigma^1 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} \tag{2.21}$$

On the S_2^3 space, we have

$$0 = i\frac{\eta_2}{2f_2}(\sigma^2 \otimes \sigma^3)\zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_2}{f_2}(1 \otimes \sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} \\ +\frac{1}{12}g_{1a}(\sigma^1 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} - \frac{i}{12}g_{3a}(\sigma^3 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} + \frac{i}{6}g_{2a}(\sigma^2 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} \tag{2.22}$$

On the S_3^3 space, we have,

$$0 = i\frac{\eta_3}{2f_3}(\sigma^3 \otimes \sigma^3)\zeta_{\eta_1,\eta_2,\eta_3} + \frac{1}{2}\frac{D_a f_3}{f_3}(1 \otimes \sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} \\ +\frac{1}{12}g_{1a}(\sigma^1 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} - \frac{i}{12}g_{2a}(\sigma^2 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} + \frac{i}{6}g_{3a}(\sigma^3 \otimes \sigma^3\sigma^a)\zeta_{\eta_1,\eta_2,\eta_3} \tag{2.23}$$

Finally, on Σ the BPS equations reduce to differential equations in ζ,

$$0 = D_a\zeta_{\eta_1,\eta_2,\eta_3} + \frac{i}{2}\hat{\omega}_a(1 \otimes \sigma^3)\zeta_{\eta_1,\eta_2,\eta_3} - \frac{i}{12}\epsilon_a{}^b(\sigma^1 \otimes 1)g_{1b}\zeta_{\eta_1,\eta_2,\eta_3} \\ -\frac{1}{12}\,\epsilon_a{}^b(\sigma^2 \otimes 1)g_{2b}\zeta_{\eta_1,\eta_2,\eta_3} - \frac{1}{12}\,\epsilon_a{}^b(\sigma^3 \otimes 1)g_{3b}\zeta_{\eta_1,\eta_2,\eta_3} \\ +\frac{1}{6}\,(\sigma^1 \otimes \sigma^3)g_{1a}\zeta_{\eta_1,\eta_2,\eta_3} - \frac{i}{6}\,(\sigma^2 \otimes \sigma^3)g_{2a}\zeta_{\eta_1,\eta_2,\eta_3} - \frac{i}{6}\,(\sigma^3 \otimes \sigma^3)g_{3a}\zeta_{\eta_1,\eta_2,\eta_3} \tag{2.24}$$

The task at hand is to simplify these equations and the goal is to solve them for ζ, as well as for the metric factors f_1, f_2, f_3, the metric ds^2_Σ, and the fluxes g_1, g_2, g_3. To do so, we begin by exhibiting discrete symmetries that will allow us to reduce the above equations for all 8 possible assignments of η_1, η_2, η_3 to the equation for just a single value $\eta_1 = \eta_2 = \eta_3 = +1$.

2.5 Discrete symmetries

The reduced BPS equations (2.21), (2.22), (2.23), and (2.24) are invariant under the following involutions,

$$\begin{aligned} &S_0 : \zeta \;\to\; i(I \otimes \sigma^3)\,\zeta \qquad && \eta_i \to -\eta_i && \\ &S_j : \zeta \;\to\; s_j(\sigma^j \otimes I)\,\zeta \qquad && \eta_i \to -(-)^{\delta_{i,j}}\eta_i \qquad && g_{ia} \to -(-)^{\delta_{i,j}}g_{ia} \end{aligned} \tag{2.25}$$

where $j = 1, 2, 3$, with $s_1 = i$, and $s_2 = s_3 = 1$. All other fields are being left invariant. The factors of i in the definitions of these transformations have been chosen so that these discrete symmetries also leave the reality condition (2.15) on ζ invariant. Taking S_0, S_1, and S_3 as the functionally independent generators of these commuting involutions, the full symmetry group consists of the eight elements $\{I,\, S_0,\, S_1,\, S_0S_1,\, S_2,\, S_0S_2,\, S_3,\, S_0S_3\}$. These 8 generators map ζ_{+++} to all eight components. Thus, it suffices to solve for ζ_{+++}.

2.6 Reducing the BPS equations to 2-component spinors

The discrete symmetries, spelled out in the preceding subsection, allow us to reduce the BPS equations for all 8 components $\zeta_{\eta_1,\eta_2,\eta_3}$ to a reduced BPS equation for just any single one of these 8 components, which we choose to be ζ_{+++}. The other seven components are then recovered by applying the involutions found above. The reality condition (2.15) on ζ_{+++} reads $(\zeta_{+++})^* = (\sigma^3 \otimes \sigma^1)\zeta_{+++}$. In a basis where $I_2 \otimes \sigma^3$ is block-diagonal, we have

$$\sigma^3 \otimes \sigma^1 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \tag{2.26}$$

and the reality condition on the 4-component spinor ζ_{+++} may be solved in terms of a 2-component complex spinor ξ and its complex conjugate ξ^*, as follows,

$$\zeta_{+++} = \begin{pmatrix} \sigma^3 \xi^* \\ \xi \end{pmatrix} \tag{2.27}$$

To recast the reduced BPS equations in terms of the 2-component spinors ξ and ξ^*, it will be convenient to make the complex structure of the tangent space of Σ explicit. This is because the complex structure that separates ξ from ξ^* is intimately intertwined with the complex structure on Σ by the reduced BPS equations.

We shall use the following conventions for the frame metric and anti-symmetric tensor,

$$\delta^{z\bar{z}} = 2 \qquad \delta_{z\bar{z}} = \frac{1}{2} \qquad \epsilon_z{}^z = -\epsilon_{\bar{z}}{}^{\bar{z}} = i \tag{2.28}$$

We introduce the following complex frame components e^z and $e^{\bar{z}}$ for Σ,

$$\begin{aligned} e^z &= (e^9 + ie^\natural)/2 \qquad & e_z &= e^9 - ie^\natural \\ e^{\bar{z}} &= (e^9 - ie^\natural)/2 \qquad & e_{\bar{z}} &= e^9 + ie^\natural \end{aligned} \tag{2.29}$$

so that $ds^2_\Sigma = 4e^z e^{\bar{z}}$, and extend this pattern for any vector or tensor, such as,

$$g_{iz} = g_{i9} - ig_{i\natural} \qquad \sigma_z = \sigma^9 - i\sigma^\natural = \sigma^1 - i\sigma^2 \tag{2.30}$$

It will also be convenient to introduce conformal coordinates w, $\bar{w}$ defined by

$$e^z = \rho(w, \bar{w})\, dw \qquad e^{\bar{z}} = \rho(w, \bar{w})\, d\bar{w} \tag{2.31}$$

where ρ is the scale factor appearing in the metric, $ds^2_\Sigma = 4\rho^2|dw|^2$. In these coordinates, the spin-connection 1-form $\hat{\omega}$ on Σ is given by

$$\hat{\omega} = -i\rho^{-1}\partial_{\bar{w}}\rho\, d\bar{w} + i\rho^{-1}\partial_w\rho\, dw \tag{2.32}$$

Expressing the reduced BPS equations (2.21), (2.22), (2.23), and (2.24), in terms of ξ, ξ^*, and conformal coordinates $w, \bar{w}$ further reduces the BPS equations. We find three purely algebraic equations in ξ and ξ^*, (recall that we have set $\eta_1 = \eta_2 = \eta_3 = +1$),

$$\begin{aligned} 0 &= -\frac{i}{2f_1}\sigma^2\xi^* + \frac{D_z f_1}{2f_1}\xi - \frac{1}{6}g_{1z}\sigma^1\xi - \frac{i}{12}g_{2z}\sigma^2\xi - \frac{i}{12}g_{3z}\sigma^3\xi \\ 0 &= -\frac{1}{2f_2}\sigma^1\xi^* + \frac{D_z f_2}{2f_2}\xi + \frac{1}{12}g_{1z}\sigma^1\xi + \frac{i}{6}g_{2z}\sigma^2\xi - \frac{i}{12}g_{3z}\sigma^3\xi \\ 0 &= +\frac{i}{2f_3}\xi^* + \frac{D_z f_3}{2f_3}\xi + \frac{1}{12}g_{1z}\sigma^1\xi - \frac{i}{12}g_{2z}\sigma^2\xi + \frac{i}{6}g_{3z}\sigma^3\xi \end{aligned} \tag{2.33}$$

and two differential equations on ξ,

$$
\begin{aligned}
0 &= D_z \xi - \frac{i}{2}\hat{\omega}_z \xi - \frac{1}{12}\, g_{1z}\sigma^1 \xi + \frac{i}{12}\, g_{2z}\sigma^2 \xi + \frac{i}{12}\, g_{3z}\sigma^3 \xi \\
0 &= D_z \xi^\dagger + \frac{i}{2}\hat{\omega}_z \xi^\dagger - \frac{1}{4}\, g_{1z}\xi^\dagger \sigma^1 - \frac{i}{4}\, g_{2z}\xi^\dagger \sigma^2 - \frac{i}{4}\, g_{3z}\xi^\dagger \sigma^3
\end{aligned} \tag{2.34}
$$

The complex conjugates of the above differential equations are given by,

$$
\begin{aligned}
0 &= D_{\bar z} \xi - \frac{i}{2}\hat{\omega}_{\bar z} \xi - \frac{1}{4}\, g_{1\bar z}\sigma^1 \xi + \frac{i}{4}\, g_{2\bar z}\sigma^2 \xi + \frac{i}{4}\, g_{3\bar z}\sigma^3 \xi \\
0 &= D_{\bar z} \xi^\dagger + \frac{i}{2}\hat{\omega}_{\bar z} \xi^\dagger - \frac{1}{12}\, g_{1\bar z}\xi^\dagger \sigma^1 - \frac{i}{12}\, g_{2\bar z}\xi^\dagger \sigma^2 - \frac{i}{12}\, g_{3\bar z}\xi^\dagger \sigma^3
\end{aligned} \tag{2.35}
$$

The reduced BPS equations (2.33) and (2.34) will constitute the basic point of departure for the construction of our solutions. In the subsequent sections, we shall systematically progress towards obtaining their complete local solutions for the cases I, II, and III defined in the Introduction, namely for boundary behavior given by $AdS_4 \times S^7$ or $AdS_7 \times S^4$.

3. Metric factors of the AdS_3 and S^3 spaces

In this section, we shall solve for the metric factors f_i as functions of bilinears in the supersymmetry parameters ξ and ξ^*. In doing so, three real integration constants c_1, c_2, c_3 will emerge. Their significance was announced in the Introduction: their absolute values $|c_1|, |c_2|$, and $|c_3|$ correspond to the inverse radii of the factor spaces AdS_3, S_2^3, and S_3^3 respectively. The constants c_1, c_2, and c_3 will obey a single relation, $c_1 + c_2 + c_3 = 0$, which implies a harmonic-like relation between the radii of the AdS_3, S_2^3, and S_3^3 factors. The solutions with 32 supersymmetries, namely $AdS_4 \times S^7$ and $AdS_7 \times S^4$, correspond to a different assignment of these constants. It will follow that for any solution with $AdS_4 \times S^7$ (respectively $AdS_7 \times S^4$) boundary asymptotics, the constants c_1, c_2, c_3 will be completely fixed as well, and that no solutions can have mixed $AdS_4 \times S^7$ and $AdS_7 \times S^4$ boundary asymptotics.

3.1 Solving for the metric factors f_i

The algebraic reduced BPS equations (2.33) involve the metric factors f_i, while the differential reduced BPS equations (2.34) do not. The metric factors f_i may be solved as bilinear functions of ξ and ξ^*. To derive this result, it will be convenient to introduce a shorthand notation for bilinears in ξ and ξ^*. A convenient basis is given by,

$$
\lambda_i = \xi^\dagger \sigma_i \xi \qquad\qquad \mu_i = \xi^t \sigma_i \xi \tag{3.1}
$$

where $i = 0, 1, 2, 3$, and we have defined $\sigma_0 \equiv I$, as usual. As a result, the combinations λ_i are real, but μ_i are generally complex. We have $\mu_2 = 0$, as well as the relations,

$$
\begin{aligned}
0 &= \lambda_0^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 \\
0 &= \mu_0^2 - \mu_1^2 - \mu_3^2
\end{aligned} \tag{3.2}
$$

From the differential equations (2.34) for ξ and $\xi^\dagger$, we deduce the following differential equations for λ_i,

$$\begin{aligned}
D_z\lambda_0 &= +\frac{1}{3}g_{1z}\lambda_1 + \frac{i}{6}g_{2z}\lambda_2 + \frac{i}{6}g_{3z}\lambda_3 \\
D_z\lambda_1 &= +\frac{1}{3}g_{1z}\lambda_0 + \frac{1}{3}g_{2z}\lambda_3 - \frac{1}{3}g_{3z}\lambda_2 \\
D_z\lambda_2 &= +\frac{i}{6}g_{1z}\lambda_3 + \frac{i}{6}g_{2z}\lambda_0 + \frac{1}{3}g_{3z}\lambda_1 \\
D_z\lambda_3 &= -\frac{i}{6}g_{1z}\lambda_2 - \frac{1}{3}g_{2z}\lambda_1 + \frac{i}{6}g_{3z}\lambda_0
\end{aligned} \tag{3.3}$$

These equations are functionally dependent, in view of the quadratic relation (3.2). Differential equations for μ_i can also be derived, but those will not be needed here.

To solve for the metric factors f_i, we begin by recasting the three algebraic reduced BPS equations (2.33) in terms of an equivalent set of two groups of equations for the bilinears of ξ. The first set is obtained by multiplying each equation of (2.33) on the left by $\xi^t\sigma^2$. This procedure has the effect of eliminating all the derivative terms in f_1, f_2, f_3, and we find,

$$\begin{aligned}
0 &= -\frac{i}{2f_1}\lambda_0 + \frac{i}{6}g_{1z}\mu_3 - \frac{i}{12}g_{2z}\mu_0 + \frac{1}{12}g_{3z}\mu_1 \\
0 &= +\frac{i}{2f_2}\lambda_3 - \frac{i}{12}g_{1z}\mu_3 + \frac{i}{6}g_{2z}\mu_0 + \frac{1}{12}g_{3z}\mu_1 \\
0 &= -\frac{i}{2f_3}\lambda_2 - \frac{i}{12}g_{1z}\mu_3 - \frac{i}{12}g_{2z}\mu_0 - \frac{1}{6}g_{3z}\mu_1
\end{aligned} \tag{3.4}$$

The second set is obtained by multiplying the first equation of (2.33) by $\xi^\dagger$, the second by $\xi^\dagger\sigma^3$ and the third by $\xi^\dagger\sigma^2$. This procedure produces combinations in terms of bilinears in ξ that are linearly independent of the ones produced in (3.4). This will guarantee that all information contained in (2.33) will have been included in the set (3.4) and the set (3.5) below. The effect of this procedure is to eliminate the first term in each of the equations of (2.33). The resulting equations are,

$$\begin{aligned}
0 &= \frac{D_z f_1}{2f_1}\lambda_0 - \frac{1}{6}g_{1z}\lambda_1 - \frac{i}{12}g_{2z}\lambda_2 - \frac{i}{12}g_{3z}\lambda_3 \\
0 &= \frac{D_z f_2}{2f_2}\lambda_3 + \frac{i}{12}g_{1z}\lambda_2 + \frac{1}{6}g_{2z}\lambda_1 - \frac{i}{12}g_{3z}\lambda_0 \\
0 &= \frac{D_z f_3}{2f_3}\lambda_2 - \frac{i}{12}g_{1z}\lambda_3 - \frac{i}{12}g_{2z}\lambda_0 - \frac{1}{6}g_{3z}\lambda_1
\end{aligned} \tag{3.5}$$

By taking suitable linear combinations of the three equations in (3.5) with those for $D_z\lambda_i$ for $i = 0, 2, 3$ in (3.3), we obtain,

$$f_1 D_z\left(\frac{\lambda_0}{f_1}\right) = f_2 D_z\left(\frac{\lambda_3}{f_2}\right) = f_3 D_z\left(\frac{\lambda_2}{f_3}\right) = 0 \tag{3.6}$$

These equations are easily integrated to give expressions for the metric factors in terms of the spinor bilinears λ_i,

$$f_1 = \frac{\lambda_0}{c_1} \qquad\qquad f_2 = -\frac{\lambda_3}{c_2} \qquad\qquad f_3 = \frac{\lambda_2}{c_3} \tag{3.7}$$

space-time	belonging to	c_1	c_2	c_3	g_{1z}	g_{2z}	g_{3z}
$AdS_4 \times S^7$	case I	-2	1	1	-3	0	0
$AdS_7 \times S^4$	case II	1	-2	1	0	$-3i$	0
$AdS_7 \times S^4$	case III	1	1	-2	0	0	$3i$

Table 2: Assignments of c_i and g_{iz} for the maximally supersymmetric cases.

where the c_i are the corresponding real integration constants. The minus sign in the definition of c_2 has been inserted in order to exhibit a higher degree of symmetry between the constants c_1, c_2, c_3, as will become clear later. Note that the absolute value $|c_i|$ may be viewed as the inverse radius of the AdS_3 or S^3 factor multiplying the corresponding bilinear λ_i.

3.2 Summary of the remaining reduced BPS equations

Using the results of (3.7), the remaining algebraic equations (3.4) may be considerably simplified. They become, after multiplication by an overall factor of 2,

$$\begin{aligned} 0 &= c_1 - \frac{1}{3} g_{1z}\mu_3 + \frac{1}{6} g_{2z}\mu_0 + \frac{i}{6} g_{3z}\mu_1 \\ 0 &= c_2 + \frac{1}{6} g_{1z}\mu_3 - \frac{1}{3} g_{2z}\mu_0 + \frac{i}{6} g_{3z}\mu_1 \\ 0 &= c_3 + \frac{1}{6} g_{1z}\mu_3 + \frac{1}{6} g_{2z}\mu_0 - \frac{i}{3} g_{3z}\mu_1 \end{aligned} \tag{3.8}$$

The sum of the three equations yields the relation on the integration constants,

$$0 = c_1 + c_2 + c_3 \tag{3.9}$$

which was announced in the Introduction. Equations (3.8) only involve ξ, ξ^*, and the reduced flux fields g_{iz}. There also remain the differential reduced BPS equations (2.34),

$$\begin{aligned} 0 &= D_z\xi - \frac{i}{2}\hat{\omega}_z\xi - \frac{1}{12}\, g_{1z}\sigma^1\xi + \frac{i}{12}\, g_{2z}\sigma^2\xi + \frac{i}{12}\, g_{3z}\sigma^3\xi \\ 0 &= D_z\xi^\dagger + \frac{i}{2}\hat{\omega}_z\xi^\dagger - \frac{1}{4}\, g_{1z}\xi^\dagger\sigma^1 - \frac{i}{4}\, g_{2z}\xi^\dagger\sigma^2 - \frac{i}{4}\, g_{3z}\xi^\dagger\sigma^3 \end{aligned} \tag{3.10}$$

3.3 Recovering the solutions with 32 supersymmetries

To understand better the significance of the constants c_i, it will be useful to recover, from the reduced BPS equations, the solutions with 32 supersymmetries, namely $AdS_7 \times S^4$, and $AdS_4 \times S^7$. Actually, there really are 3 different cases, because AdS_7 can be built up from the first S_2^3 or the second S_3^3, with corresponding assignments of the non-vanishing 4-form flux and charge. These possible cases are as given in the table below. For each case, we shall introduce local complex coordinates $w, \bar{w}$ for which $\rho = 1$, so that $ds^2_\Sigma = 4|dw|^2$. The metric factors f_i are then most conveniently expressed in terms of real coordinates x, y, defined by $w = x + iy$.

3.3.1 $AdS_4 \times S^7$ (belonging to case I)

The solution $AdS_4 \times S^7$ has the following metric factors,

$$f_1 = -\mathrm{ch}(2x) \qquad f_2 = -2\cos(y) \qquad f_3 = -2\sin(y) \tag{3.11}$$

The supersymmetry parameter is given by

$$\xi = \sqrt{2}\begin{pmatrix}\mathrm{ch}(\frac{w+3\bar{w}}{4})\\ \mathrm{sh}(\frac{w+3\bar{w}}{4})\end{pmatrix} = \sqrt{2}\begin{pmatrix}\mathrm{ch}(x-iy/2)\\ \mathrm{sh}(x-iy/2)\end{pmatrix} \tag{3.12}$$

The 10-dimensional metric is given by

$$ds^2 = \mathrm{ch}(2x)^2 ds^2_{AdS_3} + 4dx^2 + 4\cos(y)^2 ds^2_{S^3_2} + 4\sin(y)^2 ds_{S^3_3} + 4dy^2 \tag{3.13}$$

which is the standard product metric for the range $x \in \mathbf{R}$ and $0 \le y \le \pi/2$.

3.3.2 $AdS_7 \times S^4$ (belonging to case II)

The solution $AdS_7 \times S^4$ with flux on the first S^3_2 has the following metric factors,

$$f_1 = 2\mathrm{ch}(x) \qquad f_2 = \sin(2y) \qquad f_3 = 2\mathrm{sh}(x) \tag{3.14}$$

The supersymmetry parameter is given by

$$\xi = \frac{1}{\sqrt{2}}\begin{pmatrix} i\,\exp(\frac{w-3\bar{w}}{4}) - \exp(\frac{-w+3\bar{w}}{4})\\ \exp(\frac{w-3\bar{w}}{4}) - i\,\exp(\frac{-w+3\bar{w}}{4})\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i\,e^{-x/2+iy} - e^{x/2-iy}\\ e^{-x/2+iy} - ie^{x/2-iy}\end{pmatrix} \tag{3.15}$$

The 11-dimensional metric is given by

$$ds^2 = 4\mathrm{ch}(x)^2 ds^2_{AdS_3} + 4\mathrm{sh}(x)^2 ds^2_{S^3_3} + 4dx^2 + \sin(2y)^2 ds_{S^3_2} + 4dy^2 \tag{3.16}$$

which is the standard metric for the range $0 < x$ and $0 \le y \le \pi/2$.

3.3.3 $AdS_7 \times S^4$ (belonging to case III)

The solution describing $AdS_7 \times S^4$ with flux on the second S^3_3 has the following metric factors,

$$f_1 = 2\mathrm{ch}(x) \qquad f_2 = 2\mathrm{sh}(x) \qquad f_3 = \sin(2y) \tag{3.17}$$

The supersymmetry parameter is given by

$$\xi = \begin{pmatrix}\exp(\frac{+w-3\bar{w}}{4})\\ \exp(\frac{-w+3\bar{w}}{4})\end{pmatrix} = \begin{pmatrix} e^{-x/2+iy}\\ e^{+x/2-iy}\end{pmatrix} \tag{3.18}$$

The 11-dimensional metric is given by

$$ds^2 = 4\mathrm{ch}(x)^2 ds^2_{AdS_3} + 4\mathrm{sh}(x)^2 ds^2_{S^3_2} + 4dx^2 + \sin(2y)^2 ds_{S^3_3} + 4dy^2 \tag{3.19}$$

which is the standard metric for the range $0 < x$ and $0 \le y \le \pi/2$.

4. Identifying the holomorphic form κ

In this section, we shall identify, for any assignment of the constants c_1, c_2, c_3, a combination of the spinors ξ, ξ^* and the metric factor ρ, which is holomorphic on Σ as a result of the BPS equations.

From the remaining differential reduced BPS equations in (3.10), we observe the following. Viewing the reduced flux fields g_{iz} as forming part of a generalized connection on the spin bundles on Σ of which ξ and ξ^* are sections, equations (3.10) show that the connection acting on ξ^* is *minus three times* the connection acting on ξ. This suggests that holomorphic objects may be found in combinations of the type $\xi^* \otimes \xi^{\otimes 3}$. This tensor product space spans an 8-dimensional vector space, and we shall seek linear combinations in this space which are holomorphic. The exact solutions for the $AdS_4 \times S^7$ and $AdS_7 \times S^4$ cases confirm that such objects lead to holomorphic combinations for those cases.

4.1 Computing the derivatives of $\xi^* \otimes \xi^{\otimes 3}$

To carry out this calculation in practice, it is convenient to first recast (3.10) in terms of the two complex components α and β of ξ,

$$\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4.1}$$

in terms of which we have

$$\begin{aligned} \mu_0 &= \alpha^2 + \beta^2 \\ \mu_1 &= 2\alpha\beta \\ \mu_3 &= \alpha^2 - \beta^2 \end{aligned} \tag{4.2}$$

The differential equations (3.10) take the following form,

$$\begin{aligned} D_z\alpha &= +\frac{i}{2}\hat{\omega}_z\alpha + \frac{1}{12}g_{1z}\beta - \frac{1}{12}g_{2z}\beta - \frac{i}{12}g_{3z}\alpha \\ D_z\beta &= +\frac{i}{2}\hat{\omega}_z\beta + \frac{1}{12}g_{1z}\alpha + \frac{1}{12}g_{2z}\alpha + \frac{i}{12}g_{3z}\beta \\ D_z\bar{\alpha} &= -\frac{i}{2}\hat{\omega}_z\bar{\alpha} + \frac{1}{4}g_{1z}\bar{\beta} - \frac{1}{4}g_{2z}\bar{\beta} + \frac{i}{4}g_{3z}\bar{\alpha} \\ D_z\bar{\beta} &= -\frac{i}{2}\hat{\omega}_z\bar{\beta} + \frac{1}{4}g_{1z}\bar{\alpha} + \frac{1}{4}g_{2z}\bar{\alpha} - \frac{i}{4}g_{3z}\bar{\beta} \end{aligned} \tag{4.3}$$

We shall also need the derivatives of the 4 components of the tensor power $\xi^{\otimes 3}$. They may be deduced from the above equations for $D_z\alpha$ and $D_z\beta$, by direct calculation, and are given by the following expressions,

$$\begin{aligned} D_z(\alpha^3) &= \frac{3i}{2}\hat{\omega}_z\alpha^3 + \frac{1}{4}g_{1z}\alpha^2\beta - \frac{1}{4}g_{2z}\alpha^2\beta - \frac{i}{4}g_{3z}\alpha^3 \\ D_z(\alpha^2\beta) &= \frac{3i}{2}\hat{\omega}_z\alpha^2\beta + \frac{1}{12}g_{1z}(\alpha^3 + 2\alpha\beta^2) + \frac{1}{12}g_{2z}(+\alpha^3 - 2\alpha\beta^2) - \frac{i}{12}g_{3z}\alpha^2\beta \\ D_z(\alpha\beta^2) &= \frac{3i}{2}\hat{\omega}_z\alpha\beta^2 + \frac{1}{12}g_{1z}(\beta^3 + 2\alpha^2\beta) + \frac{1}{12}g_{2z}(-\beta^3 + 2\alpha^2\beta) + \frac{i}{12}g_{3z}\alpha\beta^2 \\ D_z(\beta^3) &= \frac{3i}{2}\hat{\omega}_z\beta^3 + \frac{1}{4}g_{1z}\alpha\beta^2 + \frac{1}{4}g_{2z}\alpha\beta^2 + \frac{i}{4}g_{3z}\beta^3 \end{aligned} \tag{4.4}$$

It is now straightforward to calculate the derivatives of $\xi^* \otimes \xi^{\otimes 3}$ by combining the last two equations of (4.3) with the four equations of (4.4).

4.2 Holomorphicity modulo the algebraic reduced BPS equations

We now investigate the following problem: find a linear combination, which we shall denote by $\bar{\kappa}_0$, in the 8-dimensional space $\xi^* \otimes \xi^{\otimes 3}$, whose D_z-derivative vanishes, upon the use of the algebraic remaining reduced BPS equations of (3.8), and up to factors of ρ. A general linear combination in the 8-dimensional space $\xi^* \otimes \xi^{\otimes 3}$ takes the form,

$$\begin{aligned} \bar{\kappa}_0 &= \bar{\alpha}(A_3\alpha^3 + A_2\alpha\beta^2 + A_1\alpha\beta^2 + A_0\beta^3) \\ &\quad + \bar{\beta}(B_3\alpha^3 + B_2\alpha^2\beta + B_1\alpha\beta^2 + B_0\beta^3) \end{aligned} \tag{4.5}$$

where A_i, B_i are complex constants, which remain to be determined.

From the structure of $D_z\bar{\kappa}_0$ as a function of $\alpha, \beta, \bar{\alpha}, \bar{\beta}$, it is clear that only a combination of (3.8) which is homogeneous in α and β can enter. This combination is unique, and is obtained from (3.8) by eliminating the inhomogeneous terms. To find it, we use the relation $c_1 + c_2 + c_3 = 0$ of (3.9) to recast the two remaining algebraic equations in the form,

$$\begin{aligned} g_{1z}(\alpha^2 - \beta^2) &= 2(c_1 - c_3) + 2ig_{3z}\alpha\beta \\ g_{2z}(\alpha^2 + \beta^2) &= 2(c_2 - c_3) + 2ig_{3z}\alpha\beta \end{aligned} \tag{4.6}$$

The unique homogeneous combination may be cast in the form of the equation $\mathcal{C} = 0$, where the combination $\mathcal{C}$ is defined by,

$$\mathcal{C} \equiv (c_2 - c_3)g_{1z}(\alpha^2 - \beta^2) + (c_3 - c_1)g_{2z}(\alpha^2 + \beta^2) - 2i(c_1 - c_2)g_{3z}\alpha\beta \tag{4.7}$$

Either one of the equations of (4.6) may be retained as the combination which is linearly independent of $\mathcal{C}$.

Returning to the search for $\bar{\kappa}_0$, we seek to solve the equation,

$$D_z\bar{\kappa}_0 = i\omega_z\bar{\kappa}_0 + \frac{1}{24}(\ell_0\bar{\alpha}\alpha + \ell_1\bar{\alpha}\beta + \ell_2\beta\alpha + \ell_3\bar{\beta}\beta)\,\mathcal{C} \tag{4.8}$$

for all values of g_{1z}, g_{2z}, g_{3z} *and* $\alpha, \beta, \bar{\alpha}, \bar{\beta}$, (and no longer constrained by the algebraic equations (4.6)), for some complex constants ℓ_i, $i = 0, 1, 2, 3$. The meaning of (4.8) is that a non-vanishing solution $\bar{\kappa}_0$ will be anti-holomorphic (up to a multiplicative factor of ρ) when the homogeneous algebraic equation $\mathcal{C} = 0$ is satisfied. It is straightforward to solve (4.8) for $\bar{\kappa}_0$ given by (4.5). We readily find that, for any assignment of c_1, c_2, c_3, we must have,

$$\begin{aligned} &A_0 = A_2 = \ell_0 = 0 \qquad\qquad \ell_2 = \ell_1 \\ &B_1 = B_3 = \ell_3 = 0 \end{aligned} \tag{4.9}$$

Furthermore, the non-vanishing entries of the solution are

$$\begin{aligned} A_1 = -B_2 &= -\ell_1(c_1 - c_2) \\ A_3 = -B_0 &= +\ell_1 c_3 \end{aligned} \tag{4.10}$$

Clearly, the constant ℓ_1 is just an overall multiple of the solution, which may be chosen at will. The anti-holomorphic combination $\bar\kappa$ is then given by

$$\bar\kappa = \rho\bar\kappa_0 = c_3\rho(\bar\alpha\alpha^3 - \bar\beta\beta^3) - (c_1 - c_2)\rho\alpha\beta(\bar\alpha\beta - \bar\beta\alpha) \tag{4.11}$$

up to any convenient multiplicative constant.

4.3 The differential form κ is of type $(1, 0)$

It will be useful to know that κ is a differential form on Σ of weight $(1, 0)$. To see this, it suffices to examine the coefficients of the connection $\hat\omega_z$ in the differentials of α and β; this allows us to identify forms of pure $(0, n)$ type, namely combinations which have no $\hat\omega_z$ contribution. Thus, the combinations $\sqrt{\rho}\,\alpha$, and $\sqrt{\rho}\,\beta$ must be of type $(0, n)$, for some real number n. Similarly, the combinations $\bar\alpha/\sqrt{\rho}$, and $\bar\beta/\sqrt{\rho}$ must be of type $(0, n')$, for some real number n' (which may be different from n). As a result, the combinations $\sqrt{\rho}\,\bar\alpha$, and $\sqrt{\rho}\,\bar\beta$ are of type $(n, 0)$, while the combinations $\alpha/\sqrt{\rho}$, and $\beta/\sqrt{\rho}$ are of type $(n', 0)$. Finally, we know that ρ^2 must be of type $(1, 1)$ since it is the metric. From the ratio of $\sqrt{\rho}\,\alpha$ and $\alpha/\sqrt{\rho}$, we find that ρ is of type $(-n', n)$ so that $n = -n' = 1/2$. Similarly we find, α and β are of type $(-1/4, +1/4)$, so that κ is a holomorphic form of type $(1, 0)$.

The holomorphic form κ and its complex conjugate $\bar\kappa$ may be regarded as given, on the same footing as initial value conditions. For given $\kappa, \bar\kappa$, one could then proceed to obtain the unknowns $\beta, \bar\beta$ as a function of the unknowns α and $\bar\alpha$, thereby further reducing the number of unknowns in the differential BPS equations. Since the equations for β, $\bar\beta$ as a function of α, $\bar\alpha$, and $\kappa, \bar\kappa$ are quartic, however, this way of proceeding does not appear to be useful.

5. Exact local solution of the BPS equations, case III

In this section, we shall give the complete exact local solution for the BPS equations when the constants c_i satisfy the relation $c_1 = c_2$. In view of the relation (3.9), we thus have $c_3 = -2c_1$. Without loss of generality, an overall scaling may be applied to the solution to set $c_1 = c_2 = 1$, and $c_3 = -2$. This assignment of c_i values includes the $AdS_7 \times S^4$ solution with 32 supersymmetries. It also includes all solutions that behave asymptotically as $AdS_7 \times S^4$ near any part of its boundary.

The complete general solution is obtained by using a parametrization of the reduced flux field g_{iz} adapted to the special relation $c_1 = c_2$, changing variables from $(\rho, \alpha, \beta, \bar\alpha, \bar\beta)$ to $(\rho, \kappa, \bar\kappa, \sigma, \bar\sigma)$, where κ is the holomorphic 1-form identified in the preceding section, and σ is a suitably chosen dual combination. A number of further successive changes of variables allows us to map the BPS equations onto a linear system, which can be solved exactly.

5.1 Variables adapted to $c_1 = c_2$

The algebraic reduced BPS equations, derived in (4.6), may be solved for $c_1 = c_2$ by parametrizing the fluxes g_{1z}, g_{2z}, and g_{3z} in terms of a single complex function ψ, as follows,

$$\begin{aligned} g_{1z} + g_{2z} &= 4\alpha^2\psi \\ g_{1z} - g_{2z} &= 4\beta^2\psi \\ i g_{3z} &= -\frac{3}{\alpha\beta} + \frac{\alpha^4 - \beta^4}{\alpha\beta}\psi \end{aligned} \tag{5.1}$$

The holomorphic form κ of (4.11) also simplifies in this case, and we have,

$$\kappa = \rho(\alpha\bar{\alpha}^3 - \beta\bar{\beta}^3) \tag{5.2}$$

We now think of κ as a *given* holomorphic form; its D_z differential is thus also given (up to knowledge of ρ), and not vanishing. This gives a new *algebraic* equation, which we shall now determine. To this end, we first compute,

$$\begin{aligned} \partial_w(\rho\bar{\alpha}\alpha^3) &= \rho^2\psi\alpha^2\beta^2(\alpha\bar{\beta} + \bar{\alpha}\beta) \\ \partial_w(\rho\bar{\beta}\beta^3) &= \rho^2\psi\alpha^2\beta^2(\alpha\bar{\beta} + \bar{\alpha}\beta) \\ \partial_w(\rho^{-1}\alpha\bar{\alpha}^3) &= \frac{1}{3}\psi\bar{\alpha}^2\beta^2(\bar{\alpha}\beta + 9\alpha\bar{\beta}) + \frac{2i}{3}g_{3z}\alpha\bar{\alpha}^3 \\ \partial_w(\rho^{-1}\beta\bar{\beta}^3) &= \frac{1}{3}\psi\alpha^2\bar{\beta}^2(\alpha\bar{\beta} + 9\bar{\alpha}\beta) - \frac{2i}{3}g_{3z}\beta\bar{\beta}^3 \end{aligned} \tag{5.3}$$

Subtracting the first two equations, we recover $\partial_w\bar{\kappa} = 0$. Thus, we may retain from the first two equations only their sum, and introduce a new variable for this combination,

$$\sigma = \rho(\alpha\bar{\alpha}^3 + \beta\bar{\beta}^3) \tag{5.4}$$

The key to the complete solution of the BPS equations lies in a first change of variables from $(\rho, \alpha, \beta, \bar{\alpha}, \bar{\beta})$ to the new variables, $(\rho, \kappa, \bar{\kappa}, \sigma, \bar{\sigma})$. To recover the original $\alpha, \beta, \bar{\alpha}, \bar{\beta}$, we form the combinations,

$$\begin{aligned} \sigma + \kappa &= 2\rho\alpha\bar{\alpha}^3 \\ \sigma - \kappa &= 2\rho\beta\bar{\beta}^3 \end{aligned} \tag{5.5}$$

The following quantities will be required during the course of the planned change of variables,

$$\begin{aligned} 2\rho\alpha^4 &= \frac{(\bar{\sigma} + \bar{\kappa})^{3/2}}{(\sigma + \kappa)^{1/2}} & \qquad \frac{\bar{\alpha}}{\alpha} &= \frac{(\sigma + \kappa)^{1/2}}{(\bar{\sigma} + \bar{\kappa})^{1/2}} \\ 2\rho\beta^4 &= \frac{(\bar{\sigma} - \bar{\kappa})^{3/2}}{(\sigma - \kappa)^{1/2}} & \qquad \frac{\bar{\beta}}{\beta} &= \frac{(\sigma - \kappa)^{1/2}}{(\bar{\sigma} - \bar{\kappa})^{1/2}} \end{aligned} \tag{5.6}$$

The square roots prove to be inconvenient, and may be remedied by uniformizing the problem in terms of hyperbolic functions. We define a new complex function φ instead of σ, by

$$\begin{aligned} \sigma &= \kappa\,\mathrm{ch}(2\varphi) & \qquad \sigma + \kappa &= 2\kappa(\mathrm{ch}\varphi)^2 \\ \sigma^2 - \kappa^2 &= \kappa^2(\mathrm{sh}2\varphi)^2 & \qquad \sigma - \kappa &= 2\kappa(\mathrm{sh}\varphi)^2 \end{aligned} \tag{5.7}$$

In terms of φ, we have

$$\rho\alpha^4 = \frac{\bar\kappa^{3/2}}{\kappa^{1/2}}\frac{\mathrm{ch}^3\bar\varphi}{\mathrm{ch}\varphi} \qquad \frac{\bar\alpha}{\alpha} = \frac{\kappa^{1/2}}{\bar\kappa^{1/2}}\frac{\mathrm{ch}\varphi}{\mathrm{ch}\bar\varphi}$$
$$\rho\beta^4 = \frac{\bar\kappa^{3/2}}{\kappa^{1/2}}\frac{\mathrm{sh}^3\bar\varphi}{\mathrm{sh}\varphi} \qquad \frac{\bar\beta}{\beta} = \frac{\kappa^{1/2}}{\bar\kappa^{1/2}}\frac{\mathrm{sh}\varphi}{\mathrm{sh}\bar\varphi} \tag{5.8}$$

The variables $\kappa, \bar\kappa, \varphi, \bar\varphi$ will prove to be very well-adapted to the resolution of the reduced BPS equations (3.8) and (3.10), and we now perform this change of variables there.

5.2 Changing variables in the homogeneous BPS equation

Although φ and $\bar\varphi$ will be our ultimate variables, it will be convenient to first change variables to σ. The ∂_w derivative of its complex conjugate is given by

$$\partial_w\bar\sigma = 2\rho^2\psi\alpha^2\beta^2(\alpha\bar\beta + \bar\alpha\beta) \tag{5.9}$$

In addition, the D_z-derivative equations for κ and σ may be cast in the following form,

$$\begin{aligned}\partial_w\left(\rho^{-2}\kappa\right) &= \frac{1}{3}\psi\bar\alpha^2\beta^2(\bar\alpha\beta + 9\alpha\bar\beta) - \frac{1}{3}\psi\alpha^2\bar\beta^2(\alpha\bar\beta + 9\bar\alpha\beta) + \frac{2i}{3\rho}g_{3z}\sigma\\ \partial_w\left(\rho^{-2}\sigma\right) &= \frac{1}{3}\psi\bar\alpha^2\beta^2(\bar\alpha\beta + 9\alpha\bar\beta) + \frac{1}{3}\psi\alpha^2\bar\beta^2(\alpha\bar\beta + 9\bar\alpha\beta) + \frac{2i}{3\rho}g_{3z}\kappa\end{aligned} \tag{5.10}$$

Eliminating g_{3z} between these equations, and further eliminating ψ using (5.9) gives,

$$\begin{aligned}\partial_w\left(\frac{\sigma^2-\kappa^2}{\rho^4}\right) &= \frac{\partial_w\bar\sigma}{3\rho^4}\left\{(\sigma+\kappa)\frac{\bar\beta^2}{\beta^2} + (\sigma-\kappa)\frac{\bar\alpha^2}{\alpha^2} + 8\sigma\frac{\bar\alpha\bar\beta}{\alpha\beta}\right\}\\ &\quad + \frac{8\kappa\partial_w\bar\sigma}{3\rho^4}\left(\frac{\bar\alpha\bar\beta}{\alpha\beta}\right)\left(\frac{\alpha\bar\beta - \bar\alpha\beta}{\alpha\bar\beta + \bar\alpha\beta}\right)\end{aligned} \tag{5.11}$$

Expressing all $\alpha, \beta, \bar\alpha, \bar\beta$ in terms of the known $\kappa, \bar\kappa$, and the unknown $\sigma, \bar\sigma$ gives a single ∂_w differential equation with unknowns σ and $\bar\sigma$,

$$\begin{aligned}\partial_w\ln\left(\frac{\sigma^2-\kappa^2}{\rho^4}\right) &= \frac{\partial_w\bar\sigma}{3}\left\{\frac{1}{\bar\sigma-\bar\kappa} + \frac{1}{\bar\sigma+\bar\kappa} + 8\frac{\sigma}{|\sigma^2-\kappa^2|}\right\}\\ &\quad + \frac{8\kappa\partial_w\bar\sigma}{3|\sigma^2-\kappa^2|}\left(\frac{\sigma\bar\sigma - \kappa\bar\kappa - |\sigma^2-\kappa^2|}{\sigma\bar\kappa - \kappa\bar\sigma}\right)\end{aligned} \tag{5.12}$$

Note that all ρ-dependence has been grouped on the left hand side. The first two terms of the first line on the right hand side may be integrated, using $\partial_w\bar\kappa = 0$, and the remaining terms may be regrouped,

$$\partial_w\ln\left(\frac{\sigma^2-\kappa^2}{\rho^4(\bar\sigma^2-\bar\kappa^2)^{1/3}}\right) = \frac{8\partial_w\bar\sigma}{3|\sigma^2-\kappa^2|}\left(\frac{\bar\kappa(\sigma^2-\kappa^2) - \kappa|\sigma^2-\kappa^2|}{\sigma\bar\kappa - \kappa\bar\sigma}\right) \tag{5.13}$$

We redefine ρ as follows,

$$\rho^4 = \tilde\rho^4|\sigma^2-\kappa^2|^2 \tag{5.14}$$

in terms of which we have

$$\partial_w \ln \left(\tilde{\rho}^3 (\bar{\sigma}^2 - \bar{\kappa}^2) \right) = - \frac{2\partial_w \bar{\sigma}}{|\sigma^2 - \kappa^2|} \left(\frac{\bar{\kappa}(\sigma^2 - \kappa^2) - \kappa |\sigma^2 - \kappa^2|}{\sigma \bar{\kappa} - \kappa \bar{\sigma}} \right) \tag{5.15}$$

Changing variables from σ to φ further simplifies the equation, and we get,

$$\partial_w \ln \left(\tilde{\rho}^{3/2} \mathrm{sh}(2\bar{\varphi}) \right) = -2\partial_w \bar{\varphi} \left(\frac{\mathrm{ch}(\varphi + \bar{\varphi})}{\mathrm{sh}(\varphi + \bar{\varphi})} \right) \tag{5.16}$$

5.3 Changing variables in the inhomogeneous BPS equation

The starting point is the expressions for the reduced flux fields in (5.1), and the differential equation for the ratio α/β, and for $\bar{\sigma}$,

$$D_z \ln \frac{\alpha}{\beta} = \frac{1}{12}(g_{1z} - g_{2z})\frac{\beta}{\alpha} - \frac{1}{12}(g_{1z} + g_{2z})\frac{\alpha}{\beta} - \frac{i}{6} g_{3z} \tag{5.17}$$

Eliminating the reduced flux field g_{iz} in favor of ψ using (5.1), eliminating ψ in favor of $\partial_w \bar{\sigma}$, using (5.9), and recasting the derivatives D_z in terms of derivatives with respect to the local complex coordinate w, we find,

$$\partial_w \ln \frac{\alpha^4}{\beta^4} = \frac{2\rho}{\alpha\beta} + \left(\frac{1}{\rho\alpha^4} - \frac{1}{\rho\beta^4} \right) \left(\frac{\bar{\alpha}}{\alpha} + \frac{\bar{\beta}}{\beta} \right)^{-1} \partial_w \bar{\sigma} \tag{5.18}$$

All terms, except the first one on the right hand side of this equation, are readily expressed in terms of φ, using (5.8). The missing term is handled by forming the combination,

$$\frac{\rho^8}{\alpha^8 \beta^8} = \frac{16\rho^{12}}{16\rho^4 \alpha^8 \beta^8} = 16\rho^{12} \frac{\sigma^2 - \kappa^2}{(\bar{\sigma}^2 - \bar{\kappa}^2)^3} = 16\tilde{\rho}^{12}(\sigma^2 - \kappa^2)^4 \tag{5.19}$$

To obtain the last equality above, we have made use of the definition of $\tilde{\rho}$, given by $\rho^4 = \tilde{\rho}^4 |\sigma^2 - \kappa^2|^2$. The 8-th root of this equation yields the quantity needed in the inhomogeneous reduced BPS equations. Expressing the result in terms of φ, we find,

$$\frac{\rho}{\alpha\beta} = \sqrt{2}\kappa \tilde{\rho}^{3/2} \mathrm{sh}(2\varphi) \tag{5.20}$$

Putting all together, we obtain,

$$\partial_w \ln \left(\frac{(\mathrm{ch}\bar{\varphi})^3 \mathrm{sh}\varphi}{\mathrm{ch}\varphi (\mathrm{sh}\bar{\varphi})^3} \right) = 2\sqrt{2}\kappa \tilde{\rho}^{3/2} \mathrm{sh}(2\varphi) + 16\partial_w \bar{\varphi} \frac{\mathrm{ch}\varphi (\mathrm{sh}\bar{\varphi})^3 - \mathrm{sh}\varphi (\mathrm{ch}\bar{\varphi})^3}{\mathrm{sh}(\varphi + \bar{\varphi}) \mathrm{sh}(2\bar{\varphi})} \tag{5.21}$$

A further simplified version of this equation will be presented in the summary below. It is obtained by addition to both sides the quantity $4\partial_w \ln(\mathrm{sh}\bar{\varphi}/\mathrm{ch}\bar{\varphi})$.

5.4 Summary of reduced BPS equations in $\tilde{\rho}, \varphi$ variables

The homogeneous and inhomogeneous BPS equations produce respectively,

$$\begin{aligned} \partial_w \ln \left(\tilde{\rho}^{3/2} \mathrm{sh}(2\bar{\varphi}) \right) &= -2\partial_w \bar{\varphi} \left(\frac{\mathrm{ch}(\varphi + \bar{\varphi})}{\mathrm{sh}(\varphi + \bar{\varphi})} \right) \\ \partial_w \ln \left| \mathrm{th}(\varphi) \right|^2 &= 2\sqrt{2}\kappa\, \tilde{\rho}^{3/2} \mathrm{sh}(2\varphi) + 4\partial_w \bar{\varphi} \left(\frac{\mathrm{ch}(\varphi - \bar{\varphi})}{\mathrm{sh}(\varphi + \bar{\varphi})} \right) \end{aligned} \tag{5.22}$$

For completeness, we recall the relation between ρ and $\tilde{\rho}$ with these fields,

$$\rho^2 = \tilde{\rho}^2 \kappa \bar{\kappa} |\mathrm{sh}(2\varphi)|^2 \tag{5.23}$$

5.5 Further change of variables

The system of equations (5.22) exhibits some degree of resemblance with the corresponding Type IIB equations (7.13-14) of [8]. The key difference is that, here, only a single holomorphic object κ has been exposed thus far, while in the Type IIB case, there were two. While the corresponding Type IIB equations (7.13-14) of [8] had 2 unknown real functions, the system (5.22) exhibits three unknown real functions, $\tilde{\rho}, \mathrm{Re}(\varphi), \mathrm{Im}(\varphi)$. Although this situation presents a further complication, we shall be able to solve for $\tilde{\rho}$, and map the system (5.22) to a linear partial differential equation which may be solved completely.

Given the similarity of system (5.22) with the corresponding system (7.13-14) of [8] in Type IIB, we now perform a further change of variables analogous to the one we carried out in equation (9.1) of [8] for the Type IB problem. We define the real fields μ and ϑ as follows,

$$\varphi - \bar{\varphi} = i\mu \qquad \qquad \frac{\mathrm{sh}(2\varphi)}{\mathrm{sh}(2\bar{\varphi})} = e^{2i\vartheta} \tag{5.24}$$

To convert the formulas (5.22) into these variables, we also need to have $\varphi + \bar{\varphi}$, which may be deduced from the above relations, and is given by,

$$\mathrm{th}(\varphi + \bar{\varphi}) = \frac{\mathrm{tg}(\mu)}{\mathrm{tg}(\vartheta)} \tag{5.25}$$

Other useful formulas are

$$\begin{aligned}
|\mathrm{sh}(2\varphi)|^2 &= \frac{\cos(\mu)^2 \sin(\mu)^2}{\sin(\vartheta)^2 - \sin(\mu)^2} \\
\partial(\varphi + \bar{\varphi}) &= \frac{1}{2} \frac{\sin(2\vartheta)\partial\mu - \sin(2\mu)\partial\vartheta}{\sin(\vartheta)^2 - \sin(\mu)^2} \\
\frac{\mathrm{sh}(\varphi + \bar{\varphi})}{|\mathrm{sh}(2\varphi)|} &= \frac{\cos(\vartheta)}{\cos(\mu)}
\end{aligned} \tag{5.26}$$

The range of the hyperbolic tangent function $|\mathrm{th}(\varphi + \bar{\varphi})| \leq 1$ places a restriction on the ranges of the variables μ and ϑ in view of relation (5.25), namely $|\mathrm{tg}(\mu)| \leq |\mathrm{tg}(\vartheta)|$.

5.6 The reduced BPS equations in the new variables

The first reduced BPS equation in (5.22) may be expressed as

$$\begin{aligned}
\partial_w \ln \left(\tilde{\rho}^{3/2} \mathrm{sh}(2\bar{\varphi}) \right) &= -\partial_w(\varphi + \bar{\varphi}) \left(\frac{\mathrm{ch}(\varphi + \bar{\varphi})}{\mathrm{sh}(\varphi + \bar{\varphi})} \right) + \partial_w(\varphi - \bar{\varphi}) \left(\frac{\mathrm{ch}(\varphi + \bar{\varphi})}{\mathrm{sh}(\varphi + \bar{\varphi})} \right) \\
&= -\partial_w \ln \mathrm{sh}(\varphi + \bar{\varphi}) + \frac{i\partial_w \mu}{\mathrm{th}(\varphi + \bar{\varphi})}
\end{aligned} \tag{5.27}$$

Moving the first term on the rhs to the lhs, and using (5.25) on the second term, we find,

$$\partial_w \ln \left(\tilde{\rho}^{3/2} \mathrm{sh}(2\bar{\varphi}) \mathrm{sh}(\varphi + \bar{\varphi}) \right) = +i \, \mathrm{tg}(\vartheta) \, \partial_w \ln \sin(\mu) \tag{5.28}$$

Finally, making use of (5.24), the expression inside the ln may be recast as follows,

$$\partial_w \ln \left(\tilde{\rho}^{3/2} |\mathrm{sh}(2\varphi)|^2 e^{-i\vartheta} \frac{\cos(\vartheta)}{\cos(\mu)} \right) = +i \, \mathrm{tg}(\vartheta) \, \partial_w \ln \sin(\mu) \tag{5.29}$$

Here, we have left the expression $|\mathrm{sh}(2\varphi)|^2$ unconverted, because it will be combined with the function $\tilde\rho^{3/2}$ at a later stage. The second reduced BPS equation of (5.22) may be handled analogously, and we find,

$$\partial_w \vartheta + i\partial_w \ln\sin(\mu) = 2\sqrt{2}\,\kappa\,\tilde\rho^{3/2}|\mathrm{sh}(2\varphi)|^2 e^{i\vartheta}\,\frac{\cos(\vartheta)}{\sin(2\mu)} \tag{5.30}$$

Clearly, there is a particular combination of the metric factor $\tilde\rho$ which enters in both reduced BPS equations. Therefore, we shall define a new variable $\hat\rho$ instead of $\tilde\rho$, by

$$\hat\rho^{3/2} \equiv 2\sqrt{2}\,\tilde\rho^{3/2}\frac{|\mathrm{sh}(2\varphi)|^2}{\sin(2\mu)} \tag{5.31}$$

Note that there is some arbitrariness in defining this combination. We do not include the extra factor of $\cos\vartheta$ because leaving this factor in will facilitate integrating the equations later on. In terms of the fields $\vartheta, \mu, \hat\rho$, the reduced BPS equations now take the form,

$$\begin{aligned}
\partial_w \ln\Big(\hat\rho^{3/2}e^{-i\vartheta}\cos(\vartheta)\Big) &= \Big(i\,\mathrm{tg}(\vartheta) - 1\Big)\,\partial_w \ln\sin(\mu) \\
\partial_w\vartheta + i\partial_w \ln\sin(\mu) &= \kappa\,\hat\rho^{3/2}e^{i\vartheta}\cos(\vartheta)
\end{aligned} \tag{5.32}$$

where in the first equation, we have moved a term $-\partial_w \ln\sin(\mu)$ to the right hand side.

5.7 Integrating the equation for $\hat\rho$

To integrate the reduced BPS equations of (5.32) further, we note that,

$$\begin{aligned}
i\,\mathrm{tg}(\vartheta) - 1 &= -\frac{1}{e^{i\vartheta}\cos(\vartheta)} \\
\partial_w \ln\Big(e^{-i\vartheta}\cos(\vartheta)\Big) &= \frac{-i\partial_w\vartheta}{e^{i\vartheta}\cos(\vartheta)}
\end{aligned} \tag{5.33}$$

Hence the two BPS equations of (5.32) become,

$$\begin{aligned}
\partial_w \ln\hat\rho^{3/2} &= \frac{i\partial_w\vartheta - \partial_w\ln\sin(\mu)}{e^{i\vartheta}\cos(\vartheta)} \\
i\partial_w\vartheta - \partial_w\ln\sin(\mu) &= i\kappa\,\hat\rho^{3/2}e^{i\vartheta}\cos(\vartheta)
\end{aligned} \tag{5.34}$$

Remarkably, it is now manifest that all dependence on ϑ and μ may be eliminated between these equations. Doing so, we obtain,

$$\partial_w \ln\hat\rho^{3/2} = i\,\kappa\,\hat\rho^{3/2} \tag{5.35}$$

The holomorphic differential $i\kappa$ may be written as the ∂-differential of a real harmonic function which we shall denote by h,

$$-i\kappa = \partial_w h \tag{5.36}$$

Recasting (5.35) in terms of h gives an equation which may be integrated, and we have

$$\frac{1}{\hat\rho^{3/2}} = h \tag{5.37}$$

As given by $-i\kappa = \partial_w h$, the harmonic function h is determined only up to an additive constant, which one may view as fixed by the final equation (5.37). Note that the positivity of $\hat\rho^{3/2}$ requires that $h \geq 0$ throughout Σ.

5.8 Mapping the remaining BPS equation onto a linear equation

Substituting the result (5.37) for $\hat{\rho}^{3/2}$ into either one of the BPS equations of (5.22), produces our final equation to be solved,

$$i\partial_w \vartheta - \partial_w \ln \sin(\mu) = -e^{i\vartheta} \cos(\vartheta) \partial_w \ln h \tag{5.38}$$

This is a complex first order differential equation for two real variables ϑ and μ, with the harmonic function h considered as given. The integrability condition in μ requires a second order partial differential equation for ϑ alone,

$$2\partial_{\bar{w}}\partial_w \vartheta - \partial_{\bar{w}} \Big(ie^{i\vartheta} (\cos \vartheta)\, \partial_w \ln h \Big) + \partial_w \Big(ie^{-i\vartheta} (\cos \vartheta)\, \partial_{\bar{w}} \ln h \Big) = 0 \tag{5.39}$$

This equation is of the sine-Gordon/Liouville type and is equivalent to the equation (1.1) given in the Introduction. Our task is to solve (5.38). To this end, we combine the real variables ϑ and μ into a single complex field G, defined by

$$G \equiv \sin(\mu) e^{-i\vartheta} \tag{5.40}$$

It follows that $\bar{G}/G = e^{2i\vartheta}$, which allows us to recover ϑ from G. Similarly, we have $\sin(\mu)^2 = G\bar{G}$, which allows us to recover μ from G. In terms of G, equation (5.38) becomes,

$$\partial_w G = \frac{1}{2}(G + \bar{G})\, \partial_w \ln h \tag{5.41}$$

Remarkably, we have succeeded in mapping the non-linear BPS equation (5.22) into a linear differential equation in G, and its complex conjugate equation.

5.9 Metric factors

The metric factors f_i may be readily obtained in terms of $G, \bar{G}$, and h, with the help of their expressions in terms of α and β of (3.7). The relevant quantities we need to compute are then $\alpha\bar{\alpha}$, $\beta\bar{\beta}$ and $\bar{\alpha}\beta$. Putting together the formulas from (5.8) we get the following expressions in terms of $\kappa, \bar{\kappa}, \varphi, \bar{\varphi}$, and ρ,

$$\begin{aligned} \alpha\bar{\alpha} &= \frac{|\kappa^{1/2}|}{\rho^{1/2}} |\mathrm{ch}\varphi| \\ \beta\bar{\beta} &= |\kappa^{1/2}| \rho^{-1/2} |\mathrm{sh}\varphi| \\ \bar{\alpha}\beta &= 2^{1/2} \frac{|\kappa^{1/2}|}{\rho^{1/2}} \frac{\mathrm{sh}\bar{\varphi}\, \mathrm{ch}\varphi}{|\mathrm{sh}(2\varphi)|^{1/2}} \end{aligned} \tag{5.42}$$

Next we express these quantities in terms of G, $\bar{G}$, and h. To do so, we first change variables to ϑ, μ, $\tilde{\rho}$, using (5.24) and the identities (5.25) and (5.26) for ϑ, μ, and (5.20) for $\tilde{\rho}$. Then we change variables to G, $\bar{G}$, and h using the definition of G (5.40), the definition of $\hat{\rho}$ (5.31) and the fact $\hat{\rho}^{-3/2} = h$. An ubiquitous combination entering the results is defined by

$$\begin{aligned} W^2 &= -4|G|^4 - (G - \bar{G})^2 \\ &= -4 \sin(\mu)^2 \Big(\sin(\mu)^2 - \sin(\vartheta)^2 \Big) \end{aligned} \tag{5.43}$$

In view of the inequality $|\text{tg}(\mu)| \leq |\text{tg}(\vartheta)|$, which was part of the definition of the variables μ and ϑ in section 5.5, it follows that for this range of parameters μ, ϑ, one always has $W^2 \geq 0$. The boundary is given by $W = 0$. The corresponding range for G in the complex plane such that $W^2 \geq 0$ is depicted in figure 2 of section 8. For this range of parameters adapted to the solution, W is real, and we may take it to be positive without loss of generality.

In terms of G, h, and W, we derive the following expressions for the metric factor. Using the definitions of $\tilde{\rho}$ in (5.20) and $\hat{\rho}$ in (5.31), we derive ρ in terms of G, $\bar{G}$ and h,

$$\rho^6 = \frac{|\partial_w h|^6}{16h^4}(1-|G|^2)W^2 \tag{5.44}$$

The metric factors f_1, f_2, f_3 are given by the following relations,

$$\begin{aligned} (f_1^2 - f_2^2)^3 &= 128h^2 \frac{(1-|G|^2)|G|^6}{W^4} \\ (f_1 f_2)^3 &= -4h^2 \frac{(1-|G|^2)}{W} \\ f_3^3 &= -\frac{hW}{4(1-|G|^2)} \end{aligned} \tag{5.45}$$

These relations for f_1 and f_2 may be solved more explicitly, and we have,

$$\begin{aligned} f_1^6 &= 4h^2 \frac{(1-|G|^2)}{W^4}\Big(|G-\bar{G}| + 2|G|^2\Big)^3 \\ f_2^6 &= 4h^2 \frac{(1-|G|^2)}{W^4}\Big(|G-\bar{G}| - 2|G|^2\Big)^3 \end{aligned} \tag{5.46}$$

Using $W^2 \geq 0$, we automatically have $|G-\bar{G}| \pm 2|G|^2 \geq 0$. The corresponding sign has been chosen so that $f_1^6 \geq f_2^6$, as required by the solution (3.7), the fact that $c_1 = c_2$, and the inequality $|\lambda_3| \leq \lambda_0$. Note that the product $f_1 f_2 f_3$ of the metric factors yields a simple expression solely in terms of the harmonic function h,

$$f_1 f_2 f_3 = \pm h \tag{5.47}$$

This relation was used in [33] as the definition of the harmonic coordinate h. The flux fields g_{iz} may be computed analogously, and will be given in section 9.

5.10 Calculating G and h for the $AdS_7 \times S^4$ solution

We use the expressions for the supersymmetry parameter ξ, obtained section 3.3 for the $AdS_7 \times S^4$ case, to evaluate the forms κ, σ, and φ, and we find,

$$\text{ch}(2\varphi) = -\frac{\text{ch}(2w)}{\text{sh}(2w)} \qquad\qquad \text{sh}(2\varphi) = -\frac{1}{\text{sh}(2w)} \tag{5.48}$$

These relations allow us to compute the variables μ and ϑ,

$$\sin\mu = -i\,\frac{\text{sh}(w-\bar{w})}{|\text{sh}(2w)|} \qquad\qquad e^{-i\vartheta} = \frac{|\text{sh}(2w)|}{\text{sh}(2\bar{w})} \tag{5.49}$$

and hence G and h,

$$G = -i\,\frac{\mathrm{sh}(w-\bar w)}{\mathrm{sh}(2\bar w)}$$
$$h = \frac{1}{\hat\rho^{3/2}} = -i\Big(\mathrm{ch}(2w) - \mathrm{ch}(2\bar w)\Big) \tag{5.50}$$

which is remarkably simple. For $w = x+iy$, and for the range given by $x \in \mathbf{R}$, and $0 \le y \le \pi/2$, we have $\mathrm{Im}(G) \ge 0$, so that G belongs to the range of the upper sphere in figure 2 of section 8.

6. Exact local solution of BPS equations, case II

From a physical point of view, cases II and III are mirror images of one another, obtained by interchanging the spheres S_2^3 and S_3^3. Thus, we expect the solution of case II to be substantially the same as that of case III, and to be obtained by interchanging the constants c_2 and c_3, the metric factors f_2 and f_3, the fluxes g_{2z} and g_{3z}, all while leaving the constant c_1, the metric factor f_1, and the flux g_{1z} unchanged. We shall see that this natural recipe is indeed correct, up to sign factors produced under the effects of the interchanges.

We shall establish this result as an answer to a more general question, namely, *is it possible to find a set of transformations of the reduced BPS equations, under which the BPS equations associated with one assignment of constants c_1, c_2, c_3 is mapped onto the BPS equations associated with another assignment of constants $\tilde c_1, \tilde c_2, \tilde c_3$?* It is assumed, of course, that the relations $c_1+c_2+c_3=0$, and $\tilde c_1+\tilde c_2+\tilde c_3=0$ hold, since these equations form an integral part of the BPS equations. The map between case II and case III is a simple special case of this question.

It would be difficult to try and address this question in all generality, since it is difficult to investigate the behavior of the BPS equations under a completely general transformation. Thus, in the present paper, we shall restrict attention to *the most general constant transformation which acts linearly on the reduced supersymmetry parameters ξ.*

In the preceding sections, the BPS equations have been reduced to a set of first order differential equations (3.10) on the supersymmetry parameter,

$$0 = D_z\xi - \frac{i}{2}\hat\omega_z\xi - \frac{1}{12}\,g_{1z}\sigma^1\xi + \frac{i}{12}\,g_{2z}\sigma^2\xi + \frac{i}{12}\,g_{3z}\sigma^3\xi$$
$$0 = D_{\bar z}\xi - \frac{i}{2}\hat\omega_{\bar z}\xi - \frac{1}{4}\,g_{1\bar z}\sigma^1\xi + \frac{i}{4}\,g_{2\bar z}\sigma^2\xi + \frac{i}{4}\,g_{3\bar z}\sigma^3\xi \tag{6.1}$$

and a subsidiary set of algebraic equations (3.8), equivalent to,

$$g_{1z}\mu_3 = 2c_1 - 2c_3 + ig_{3z}\mu_1$$
$$g_{2z}\mu_0 = 2c_2 - 2c_3 + ig_{3z}\mu_1 \tag{6.2}$$

The g_{iz} are components of real tensors g_{ia}, and satisfy $g_{i\bar z} = (g_{iz})^*$ for $i=1,2,3$. The bilinears μ_i are defined by $\mu_i = \xi^t\sigma_i\xi$ with $i=0,1,2,3$. The μ_i automatically satisfy $\mu_0^2 - \mu_1^2 - \mu_3^2 = 0$. The real constants c_i must be non-zero and are related by $c_1+c_2+c_3=$

0. The differential equations may be recast in a more compact form by introducing the following combination,

$$\begin{aligned}\mathcal{G}_z &= g_{1z}\sigma^1 - ig_{2z}\sigma^2 - ig_{3z}\sigma^3 \\ -\sigma^3(\mathcal{G}_z)^*\sigma^3 &= g_{1\bar{z}}\sigma^1 - ig_{2\bar{z}}\sigma^2 - ig_{3\bar{z}}\sigma^3\end{aligned} \tag{6.3}$$

As a result, the differential BPS equations (6.1) become,

$$\begin{aligned}0 &= D_z\xi - \frac{i}{2}\hat{\omega}_z\xi - \frac{1}{12}\mathcal{G}_z\xi \\ 0 &= D_{\bar{z}}\xi - \frac{i}{2}\hat{\omega}_{\bar{z}}\xi + \frac{1}{4}\sigma^3(\mathcal{G}_z)^*\sigma^3\xi\end{aligned} \tag{6.4}$$

The combinations $(\mathcal{G}_z, \mathcal{G}_{\bar{z}})$ may be thought of as the components of a generalized connection 1-form, coupling to ξ in a left-right asymmetric way.

6.1 Symmetries of the differential BPS equations

We begin by seeking constant (i.e. Σ-independent) linear transformations on ξ, and suitable associated transformations on g_i (which will not be linear) such that the differential BPS equations are left invariant, irrespective of the algebraic equations. The next step will then be to require that also the algebraic reduced BPS equations are invariant.

A general constant complex transformation S, which acts on ξ and $\mathcal{G}_z$ by

$$\begin{aligned}\xi &= S\tilde{\xi} \\ \mathcal{G}_z &= S\tilde{\mathcal{G}}_z S^{-1}\end{aligned} \tag{6.5}$$

will leave the first differential equation of (6.4) invariant. The overall multiplicative factor in S acts as a constant $\mathrm{U}(1)$ phase and a real scale factor on ξ. The phase symmetry guarantees that ξ and ξ^* do not mix. The second differential equation will also be invariant under (6.5) provided that we have

$$\sigma^3(\mathcal{G}_z)^*\sigma^3 = S\sigma^3(\tilde{\mathcal{G}}_z)^*\sigma^3 S^{-1} \tag{6.6}$$

Taking the complex conjugate of this equation, and eliminating $\tilde{\mathcal{G}}_z$ between the resulting equation and the second equation of (6.5) gives the following requirement,

$$\mathcal{G}_z = \sigma^3 S^*\sigma^3 S^{-1}\mathcal{G}_z\left(\sigma^3 S^*\sigma^3 S^{-1}\right)^{-1} \tag{6.7}$$

In other words, the matrix $\sigma^3 S^*\sigma^3 S^{-1}$ must commute with $\mathcal{G}_z$. For a general solution to the BPS equations with 16 supersymmetries, the flux field $\mathcal{G}_z$ will take on generic values.[4] By Shur's lemma, we are led to require $\sigma^3 S^*\sigma^3 S^{-1} = t^2 I$, where t is a constant complex parameter. Taking the determinant on both sides of (6.7) informs us that $|t| = 1$. Since the phase transformation was a symmetry of the differential BPS equations, we may choose $t = 1$ without loss of generality. This gives rise to our final equation for S,

$$S = \sigma^3 S^*\sigma^3 \tag{6.8}$$

[4]Note that for the solutions with 32 supersymmetries, the fluxes *do not take on generic values*, since only a single one of the fluxes g_{1z}, g_{2z}, g_{3z} will be non-vanishing.

The complete solution for S is given as follows,

$$S = \begin{pmatrix} a & ib \\ ic & d \end{pmatrix} \qquad\qquad a,b,c,d \in \mathbf{R} \tag{6.9}$$

The determinant of S does not act on $\mathcal{G}_z$ and acts on ξ as a scale factor under which the differential equations are invariant. In the algebraic equations, the scale factor will scale all c_i in the same manner, and so is also a symmetry. As a result, the scale factor in S is irrelevant, and we set $\det(S) = 1$ without loss of generality. Thus, we have $ad+bc = 1$. The set of these matrices forms a group isomorphic to $\mathrm{SL}(2, \mathbf{R})$, under which the differential BPS equations (6.1) and (6.4) are invariant.

6.2 No continuous symmetries of the full set of BPS equations

To investigate the behavior under S of the algebraic equations, we need the behavior of g_{iz} and of μ_i under S. They may be obtained from (6.5), and are given by

$$\begin{aligned} \tilde{\mathcal{G}}_z &= S^{-1}\mathcal{G}_z S \\ \tilde{\mu}_i &= \tilde{\xi}^t \sigma_i \tilde{\xi} = \xi^t (S^t)^{-1}\sigma_i S^{-1}\xi \end{aligned} \tag{6.10}$$

It is convenient to write out these transformations in components, and we find,

$$\begin{pmatrix} \tilde{g}_{1z} \\ \tilde{g}_{2z} \\ \tilde{g}_{3z} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \begin{pmatrix} g_{1z} \\ g_{2z} \\ g_{3z} \end{pmatrix} \tag{6.11}$$

where the entries u_{ij} are given in terms of the entries a, b, c, d of S by,

$$\begin{aligned} 2u_{11} &= a^2+b^2+c^2+d^2 & u_{13} &= -ac+bd & u_{31} &= -ab+cd \\ 2u_{12} &= a^2+b^2-c^2-d^2 & u_{23} &= -ac-bd & u_{32} &= ab+cd \\ 2u_{21} &= a^2-b^2+c^2-d^2 & & & u_{33} &= ad-bc \\ 2u_{22} &= a^2-b^2-c^2+d^2 & & & & \end{aligned} \tag{6.12}$$

Note that, although S is complex, the action of S on the flux fields g_{1z}, g_{2z}, g_{3z} is real. Similarly, the bilinears μ_i transform in a related manner,

$$\begin{pmatrix} \tilde{\mu}_3 \\ \tilde{\mu}_0 \\ \tilde{\mu}_1 \end{pmatrix} = \begin{pmatrix} u_{11} & -u_{12} & -iu_{13} \\ -u_{21} & u_{22} & iu_{23} \\ -iu_{31} & -iu_{32} & u_{33} \end{pmatrix} \begin{pmatrix} \mu_3 \\ \mu_0 \\ \mu_1 \end{pmatrix} \tag{6.13}$$

It remains to analyze the algebraic equations. Assuming that the original configuration satisfies the algebraic equations,

$$\begin{aligned} g_{1z}\mu_3 - ig_{3z}\mu_1 &= 2c_1 - 2c_3 \\ g_{2z}\mu_0 - ig_{3z}\mu_1 &= 2c_2 - 2c_3 \end{aligned} \tag{6.14}$$

for some assignment of c_i satisfying $c_1 + c_2 + c_3 = 0$, we shall require that the transformed configuration also satisfy the algebraic equations, but possibly for a different set of $\tilde{c}_i$,

$$\begin{aligned} \tilde{g}_{1z}\tilde{\mu}_3 - i\tilde{g}_{3z}\tilde{\mu}_1 &= 2\tilde{c}_1 - 2\tilde{c}_3 \\ \tilde{g}_{2z}\tilde{\mu}_0 - i\tilde{g}_{3z}\tilde{\mu}_1 &= 2\tilde{c}_2 - 2\tilde{c}_3 \end{aligned} \tag{6.15}$$

Computing the left hand side of the first line, we get

$$\begin{aligned} \tilde{g}_{1z}\tilde{\mu}_3 - i\tilde{g}_{3z}\tilde{\mu}_1 = & \, g_{1z}\mu_3(u_{11}^2 - u_{31}^2) + g_{2z}\mu_0(-u_{12}^2 - u_{32}^2) - ig_{3z}\mu_1(u_{13}^2 + u_{33}^2) \\ & - g_{1z}\mu_0(u_{11}u_{12} + u_{31}u_{32}) - ig_{1z}\mu_1(u_{11}u_{13} + u_{31}u_{33}) \\ & + g_{2z}\mu_3(u_{11}u_{12} - u_{31}u_{32}) - ig_{2z}\mu_1(u_{12}u_{13} + u_{32}u_{33}) \\ & + g_{3z}\mu_3(u_{11}u_{13} - u_{31}u_{33}) - g_{3z}\mu_0(u_{12}u_{13} + u_{32}u_{33}) \end{aligned} \tag{6.16}$$

The last six terms do not fit into the patterns of the algebraic equations, and thus should vanish. It is immediate that this requires

$$u_{11}u_{12} = u_{31}u_{32} = u_{11}u_{13} = u_{31}u_{33} = u_{12}u_{13} + u_{32}u_{33} = 0 \tag{6.17}$$

Since from its very expression in (6.12), we have $u_{11} \neq 0$, it must be that $u_{12} = u_{13} = 0$. These relations imply the following relations on the parameters a, b, c, d of S,

$$a^2 + b^2 - c^2 - d^2 = ac - bd = 0 \tag{6.18}$$

Given that also $ad + bc = 1$, these conditions are immediately solved by the following parametrization in terms of a single real angle θ,

$$\begin{aligned} a = d &= \cos\theta \\ b = c &= \sin\theta \end{aligned} \tag{6.19}$$

for any $0 \leq \theta < 2\pi$. As a result, we have $u_{11} = 1$, and $u_{12} = u_{13} = u_{21} = u_{31} = 0$, as well as,

$$\begin{aligned} u_{22} = u_{33} &= \cos 2\theta \\ -u_{23} = u_{32} &= \sin 2\theta \end{aligned} \tag{6.20}$$

Condition (6.17), however, implies that we must have $\sin 4\theta = 0$. Thus, the parameter θ can take only discrete values, and we conclude that the reduced BPS equations admit no continuous linear symmetries.

6.3 Discrete symmetries of the full BPS equations

The only nontrivial transformations left to investigate correspond to values of θ such that $\sin 4\theta = 0$. Letting $\theta \to \theta + \pi$ reverses the sign of a, b, c, d, and thus leaves all u_{ij} invariant; this transformation simply reverses the sign of ξ, leaving the fluxes g_{1z}, g_{2z}, g_{3z} unchanged. This transformation is trivial. The case $\theta = 0$ is simply the identity, and is also trivial, leaving only the following possible values, $\theta = \pi/4, \pi/2, 3\pi/4$ as non-trivial cases.

JHEP08(2008)028

The case $\theta = \pi/2$ yields $u_{11} = 1$, and $u_{22} = u_{33} = -1$, so that $g_{2z}, g_{3z}, \mu_0, \mu_1$ change sign, but g_{1z} and μ_3 are unchanged. This transformation leaves the values of c_1, c_2, c_3 unchanged as well, and is thus also essentially trivial.

The only non-trivial case left is $\theta = \pi/4$. (The case $\theta = 3\pi/4$ is equivalent to the case $\theta = \pi/4$ combined with the sign changes on the fluxes and bilinears produced by $\theta = \pi/2$. Therefore, the case $\theta = 3\pi/4$ need not be considered separately.) The case $\theta = \pi/4$ corresponds to interchanging the two S^3 spheres as follows,

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \qquad\qquad u_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}_{ij} \tag{6.21}$$

This produces the following transformation of the g_{iz}

$$\tilde{g}_{1z} = g_{1z} \qquad\qquad \tilde{g}_{2z} = -g_{3z} \qquad\qquad \tilde{g}_{3z} = g_{2z} \tag{6.22}$$

In order for (6.14) to remain invariant, the metric constants must transform as

$$\tilde{c}_1 = c_1 \qquad\qquad \tilde{c}_2 = c_3 \qquad\qquad \tilde{c}_3 = c_2 \tag{6.23}$$

Finally using the expression for the metric factors in terms of the spinor bi-linears (3.7), one may also work out the transformation of the metric factors

$$\tilde{f}_1 = f_1 \qquad\qquad \tilde{f}_2 = -f_3 \qquad\qquad \tilde{f}_3 = f_2 \tag{6.24}$$

We see that the effect of swapping c_2 with c_3 amounts to swapping the two S^3 spheres and physically corresponds to the same geometry.

7. Exact local solution of BPS equations, case I

Case I is physically different from cases II and III. In principle, the geometry of case I could be obtained from the geometry of cases II or III by a double analytic continuation in which the AdS_3 of case III is mapped into one of the S^3 of case I and one of the S^3 of case III is mapped into the AdS_3 of case I. This analytic continuation is very delicate, however, and we have found it difficult to carry it out with absolute confidence. For this reason, we shall derive the solution for case I again from first principles, just as we did for case III.

For case I, we have $c_2 = c_3$, and thus $c_1 = -2c_2$. By an overall rescaling of the case I geometry, we may choose $c_2 = c_3 = 1$ and thus $c_1 = -2$. The holomorphic 1-form κ, given by (4.11) up to an overall multiplicative constant, will be normalized as follows,

$$\kappa = \rho(\alpha + \beta)(\bar{\alpha} - \bar{\beta})^3 + \rho(\alpha - \beta)(\bar{\alpha} + \bar{\beta})^3 \tag{7.1}$$

The structure of κ in terms of α and β suggests carrying out the following change of variables,

$$\begin{aligned} a &\equiv \alpha + \beta \\ b &\equiv \alpha - \beta \end{aligned} \tag{7.2}$$

Expressing the reduced BPS equations (4.3) in terms of $a, b, \bar{a}, \bar{b}$, the corresponding differential equations are given as follows,

$$\begin{aligned} D_z a &= +\frac{i}{2}\hat{\omega}_z a + \frac{1}{12} g_{1z} a + \frac{1}{12} g_{2z} b - \frac{i}{12} g_{3z} b \\ D_z b &= +\frac{i}{2}\hat{\omega}_z b - \frac{1}{12} g_{1z} b - \frac{1}{12} g_{2z} a - \frac{i}{12} g_{3z} a \\ D_z \bar{a} &= -\frac{i}{2}\hat{\omega}_z \bar{a} + \frac{1}{4} g_{1z} \bar{a} + \frac{1}{4} g_{2z} \bar{b} + \frac{i}{4} g_{3z} \bar{b} \\ D_z \bar{b} &= -\frac{i}{2}\hat{\omega}_z \bar{b} - \frac{1}{4} g_{1z} \bar{b} - \frac{1}{4} g_{2z} \bar{a} + \frac{i}{4} g_{3z} \bar{a} \end{aligned} \tag{7.3}$$

while the corresponding algebraic equations are

$$\begin{aligned} g_{2z}(a^2+b^2) - i g_{3z}(a^2-b^2) &= 0 \\ 2 g_{1z} ab - i g_{3z}(a^2-b^2) &= 12 \end{aligned} \tag{7.4}$$

7.1 Variables adapted to $c_2 = c_3$

The first equation of (7.4) is homogeneous, and allows for a convenient parametrization of the flux field in terms of a single complex form ψ,

$$\begin{aligned} g_{2z} + i g_{3z} &= +4a^2\psi \\ g_{2z} - i g_{3z} &= -4b^2\psi \end{aligned} \tag{7.5}$$

In terms of a and b, the holomorphic 1-form κ and its conjugate σ, take on simple forms,

$$\begin{aligned} \kappa &= \rho(a\bar{b}^3 + b\bar{a}^3) & \qquad \bar{\kappa} &= \rho(\bar{a}b^3 + \bar{b}a^3) \\ \sigma &= \rho(a\bar{b}^3 - b\bar{a}^3) & \qquad \bar{\sigma} &= \rho(\bar{a}b^3 - \bar{b}a^3) \end{aligned} \tag{7.6}$$

The four differential reduced BPS equations of (7.3) are equivalent to the following two equations for $\bar{\kappa}, \bar{\sigma}$,

$$\begin{aligned} \partial_w \bar{\kappa} &= 0 \\ \partial_w \bar{\sigma} &= -2\rho^2 \psi a^2 b^2 (a\bar{a} - b\bar{b}) \end{aligned} \tag{7.7}$$

and the following two equations for κ, σ,

$$\begin{aligned} \partial_w\left(\frac{\kappa}{\rho^2}\right) &= -\frac{2}{3} g_{1z}\frac{\sigma}{\rho} - \frac{1}{3}\psi a^2\bar{a}^2(a\bar{a} - 9b\bar{b}) + \frac{1}{3}\psi b^2\bar{b}^2(9a\bar{a} - b\bar{b}) \\ \partial_w\left(\frac{\sigma}{\rho^2}\right) &= -\frac{2}{3} g_{1z}\frac{\kappa}{\rho} + \frac{1}{3}\psi a^2\bar{a}^2(a\bar{a} - 9b\bar{b}) + \frac{1}{3}\psi b^2\bar{b}^2(9a\bar{a} - b\bar{b}) \end{aligned} \tag{7.8}$$

The advantage of the variables $\kappa, \bar{\kappa}, \sigma, \bar{\sigma}$ is that κ and $\bar{\kappa}$ should now be viewed as a *given* holomorphic forms, thus reducing the number of unknowns to only two, namely $\sigma, \bar{\sigma}$. In analogy with the steps followed in the solution of case III, we introduce the combination,

$$\partial_w\left(\frac{\sigma^2 - \kappa^2}{2\rho^4}\right) = \frac{\psi}{3\rho^2}\Big[a^2\bar{a}^2(a\bar{a} - 9b\bar{b})(\sigma+\kappa) + b^2\bar{b}^2(9a\bar{a} - b\bar{b})(\sigma - \kappa)\Big] \tag{7.9}$$

Next, we eliminate ψ, $a, b, \bar{a}, \bar{b}$ in favor of $\kappa, \bar{\kappa}, \sigma, \bar{\sigma}$ throughout. To do so, it is useful to have the following formulas,

$$\frac{\bar{a}^2}{b^2} = -\frac{\sigma - \kappa}{\bar{\sigma} + \bar{\kappa}} \qquad \frac{a\bar{a}}{b\bar{b}} = \left|\frac{\sigma - \kappa}{\sigma + \kappa}\right| \qquad \frac{\bar{a}\bar{b}}{ab} = -\frac{\sigma^2 - \kappa^2}{|\sigma^2 - \kappa^2|} \tag{7.10}$$

As for case III, it is advantageous to introduce a rescaled metric factor $\tilde{\rho}^2$ on Σ, defined by

$$\rho^4 = \tilde{\rho}^4 |\sigma^2 - \kappa^2|^2 \tag{7.11}$$

Upon carrying out the above elimination, we find,

$$\partial_w \ln \left(\tilde{\rho}^3 (\bar{\sigma}^2 - \bar{\kappa}^2) \right) = \frac{2\partial_w \bar{\sigma}}{|\sigma^2 - \kappa^2|} \left(\frac{\bar{\kappa}(\sigma^2 - \kappa^2) - \kappa|\sigma^2 - \kappa^2|}{\sigma\bar{\kappa} + \kappa\bar{\sigma}} \right) \tag{7.12}$$

Finally, using the further change of variables of (5.7) to $\varphi, \bar{\varphi}$, we find,

$$\partial_w \ln \left(\tilde{\rho}^{3/2} \mathrm{sh}(2\bar{\varphi}) \right) = 2\partial_w \bar{\varphi} \, \frac{\mathrm{sh}(\varphi - \bar{\varphi})}{\mathrm{ch}(\varphi - \bar{\varphi})} \tag{7.13}$$

We now perform the corresponding changes of variables in the inhomogeneous algebraic reduced BPS equation of (7.4) as well. The starting point is the second algebraic equation in (7.4), expressed in terms of ψ,

$$g_{1z} = \frac{6}{ab} + \frac{a^4 - b^4}{ab} \psi \tag{7.14}$$

as well as the differential relation for the ratio a/b,

$$\partial_w \ln \frac{a}{b} = \frac{1}{ab} + \frac{a^4 - b^4}{2ab} \psi \tag{7.15}$$

Eliminating ψ and $a, b, \bar{a}, \bar{b}$ in favor of $\varphi, \bar{\varphi}$ in all terms requires the computation of $\rho/(ab)$. As for case III, this term is first computed to the 8-th power, and we find,

$$\frac{\rho^8}{a^8 b^8} = 16\kappa^8 \tilde{\rho}^{12} \mathrm{sh}(2\varphi)^8 \tag{7.16}$$

Taking its 8-th root will generally introduce an 8-th root of unity ν,

$$\frac{\rho}{ab} = \sqrt{2}\nu\kappa\tilde{\rho}^{3/2} \mathrm{sh}(2\varphi) \tag{7.17}$$

This yields the following equation,

$$\partial_w \ln \left(\frac{\mathrm{th}\varphi}{\mathrm{th}\bar{\varphi}} \right) = 4\sqrt{2}\nu\kappa\tilde{\rho}^{3/2} \mathrm{sh}(2\varphi) + 4\partial_w \bar{\varphi} \, \frac{\mathrm{sh}(\varphi + \bar{\varphi})}{\mathrm{ch}(\varphi - \bar{\varphi})} \tag{7.18}$$

The 8-th root of unity may be further constrained by the following considerations. Taking the square of (7.17), and dividing the result by the first formula in (7.10), we find a positive combination, and this requires that $\nu^2 = -1$, so that ν is actually a 4-th root of unity only, and equal to $\nu = \pm i$.

JHEP08(2008)028

7.2 Further change of variables

The natural variables ϑ, μ for this case are

$$\varphi + \bar{\varphi} = \mu \qquad\qquad \frac{\mathrm{sh}(2\varphi)}{\mathrm{sh}(2\bar{\varphi})} = e^{2i\vartheta} \tag{7.19}$$

In case I, (in contrast with case III) there are no restrictions on the ranges of μ and ϑ, which can take any real values. The combination $\varphi - \bar{\varphi}$ is then given by

$$\mathrm{th}(\varphi - \bar{\varphi}) = i\,\mathrm{tg}(\vartheta)\mathrm{th}(\mu) \tag{7.20}$$

The following formulas will be useful,

$$\begin{aligned}
|\mathrm{sh}(2\varphi)|^2 &= \frac{\mathrm{ch}(\mu)^2\mathrm{sh}(\mu)^2}{\cos(\vartheta)^2 + \mathrm{sh}(\mu)^2} \\
\partial(\varphi - \bar{\varphi}) &= \frac{i}{2}\,\frac{\sin(2\vartheta)\partial\mu + \mathrm{sh}(2\mu)\partial\vartheta}{\cos(\vartheta)^2 + \mathrm{sh}(\mu)^2} \\
\frac{\mathrm{sh}(\varphi - \bar{\varphi})}{|\mathrm{sh}(2\varphi)|} &= \pm i\,\frac{\sin(\vartheta)}{\mathrm{ch}(\mu)}
\end{aligned} \tag{7.21}$$

As in case III, it will be useful to introduce a rescaled metric factor $\hat{\rho}^{3/2}$, defined by

$$\hat{\rho}^{3/2} = \frac{4\sqrt{2}|\mathrm{sh}(2\varphi)|^2}{\mathrm{sh}(2\mu)}\,\tilde{\rho}^{3/2} \tag{7.22}$$

In terms of κ, $\hat{\rho}, \mu$ and ϑ, the reduced BPS equations become,

$$\begin{aligned}
\partial_w\vartheta + i\partial_w \ln \mathrm{ch}(\mu) &= -i\nu\hat{\rho}^{3/2}e^{i\vartheta}\cos(\vartheta) \\
\partial_w \ln\Big(\hat{\rho}^{3/2}e^{-i\vartheta}\cos(\vartheta)\Big) &= \Big(i\,\mathrm{tg}(\vartheta) - 1\Big)\partial_w \ln \mathrm{ch}(\mu)
\end{aligned} \tag{7.23}$$

Expressing the holomorphic 1-form κ as the $(1,0)$-differential of a real harmonic function h,

$$\kappa = \nu\partial_w h \tag{7.24}$$

the equation for $\hat{\rho}$ may be integrated,

$$\frac{1}{\hat{\rho}^{3/2}} = h \tag{7.25}$$

To integrate the remaining equation, we introduce yet one more change of variables,

$$G \equiv \mathrm{ch}(\mu)e^{-i\vartheta} \tag{7.26}$$

By its very construction, and the fact that μ and ϑ are real, we must have $|G| \geq 1$. In terms of G, the remaining equation of (7.23) takes the form,

$$\partial_w G = \frac{1}{2}(G + \bar{G})\partial_w \ln h \tag{7.27}$$

This equation is identical to the one encountered for case III and may be solved by the same methods. We shall do so explicitly for both cases in section 8.

7.3 Metric factors

For case I, the metric factors f_1, f_2, f_3 are given in terms of a and b by

$$\begin{aligned} f_1 &= -\frac{1}{2}\lambda_0 = -\frac{1}{4}(a\bar{a} + b\bar{b}) \\ f_2 &= -\lambda_3 = -\frac{1}{2}(a\bar{b} + \bar{a}b) \\ f_3 &= +\lambda_2 = \frac{i}{2}(\bar{a}b - a\bar{b}) \end{aligned} \tag{7.28}$$

It is convenient to first work out the form of the Σ-metric factor ρ, by converting ρ to $\hat{\rho}$, the latter being known directly in terms of the harmonic function h. The following combination

$$W^2 \equiv 4|G|^4 + (G - \bar{G})^2 \tag{7.29}$$

enters ubiquitously. As a result of the defining range $|G| \geq 1$ of G, it readily follows that within this range, we automatically have $W^2 \geq 0$, so that W is real, and we shall take it to be positive throughout. To calculate ρ, a helpful formula is as follows,

$$\cos(\vartheta)^2 + \text{sh}(\mu)^2 = \frac{W^2}{4|G|^2} \tag{7.30}$$

The result is most easily expressed as a formula for ρ^6, given by,

$$\rho^6 = \frac{|\partial_w h|^6}{(4h)^4}(|G|^2 - 1)W^2 \tag{7.31}$$

The metric factors f_1, f_2, f_3 are given by the following expressions,

$$\begin{aligned} f_1^3 &= \frac{hW}{16(|G|^2 - 1)} \\ (f_2^2 + f_3^2)^3 &= \frac{16h^2|G|^6(|G|^2 - 1)}{W^4} \\ f_2^3 f_3^3 &= \frac{h^2(|G|^2 - 1)}{4W} \end{aligned} \tag{7.32}$$

Solving the above relations for f_2^6 and f_3^6, we find,

$$\begin{aligned} f_2^6 &= \frac{2h^2(|G|^2 - 1)}{W^4}\Big(|G|^2 \pm \frac{1}{2}|G - \bar{G}|\Big)^3 \\ f_3^6 &= \frac{2h^2(|G|^2 - 1)}{W^4}\Big(|G|^2 \mp \frac{1}{2}|G - \bar{G}|\Big)^3 \end{aligned} \tag{7.33}$$

The correlated sign choices under the 3-rd powers in the above formulas are reversed under the interchange of the spheres S_2^3 and S_3^3. The product of the metric factors is again proportional to h, and we have $f_1 f_2 f_3 = \pm h/4$.

JHEP08(2008)028

7.4 Calculating G and h for the $AdS_4 \times S^7$ solution

As a check, using the above changes of variables, one may evaluate G and h for the $AdS_4 \times S_7$ solution with 32 supersymmetries. The starting point may be taken to be the definition of the function φ in terms of w, given in subsection 3.3,

$$\mathrm{sh}(2\varphi) = \frac{i}{\mathrm{ch}(2w)} \tag{7.34}$$

All other functions needed may be computed from this correspondence, and we find,

$$\mathrm{ch}(\mu) = \frac{\mathrm{ch}(w+\bar{w})}{|\mathrm{ch}(2w)|} \qquad\qquad e^{-i\vartheta} = i\frac{|\mathrm{ch}(2w)|}{\mathrm{ch}(2\bar{w})} \tag{7.35}$$

Using the relation between $\hat{\rho}$ and h, and the relations between $\hat{\rho}, \tilde{\rho}$, with $\rho = 1$, we find,

$$\begin{aligned} G &= i\frac{\mathrm{ch}(w+\bar{w})}{\mathrm{ch}(2\bar{w})} \\ h &= 4i(\mathrm{sh}(2w) - \mathrm{sh}(2\bar{w})) \end{aligned} \tag{7.36}$$

For the range $w = x + iy$, with $x \in \mathbf{R}$, and $0 \leq y \leq \pi/2$, the function G sweeps through the entire range $|G| \geq 1$ of figure 2 in section 8.

8. General solution of the linear equation for G

For cases I, II, and III, the harmonic function h and the complex function G satisfy the same complex linear partial differential equation on Σ,

$$\partial_w G = \frac{1}{2}(G + \bar{G})\partial_w \ln h \tag{8.1}$$

The key difference between case I on the one hand, and cases II and III on the other hand, is the allowed range of the function G. These ranges are defined by

$$\begin{array}{ll} \text{case I} & |G| \geq 1 \\ \text{cases II and III} & W^2 = -4|G|^4 - (G - \bar{G})^2 \geq 0 \end{array} \tag{8.2}$$

and are depicted in figure 2 below.

Equation (8.1) for G is manifestly covariant under conformal reparametrizations of the local conformal coordinate w. We shall take advantage of this invariance to choose conformal coordinates $u = r + ix$ adapted to the harmonic function h, and defined as follows,

$$\begin{aligned} h &= r & \qquad 2\partial_u &= \partial_r - i\partial_x \\ \tilde{h} &= x & \qquad 2\partial_{\bar{u}} &= \partial_r + i\partial_x \end{aligned} \tag{8.3}$$

Here, we have introduced also the harmonic function $\tilde{h}$ dual to h, so that $\partial_{\bar{u}}(h + i\tilde{h}) = 0$. Recall that since $r = h = \hat{\rho}^{-3/2}$, only the domain $r \geq 0$ is allowed for regular real solutions. It is for this reason that we have used the notation r, typical of a radial variable for h.

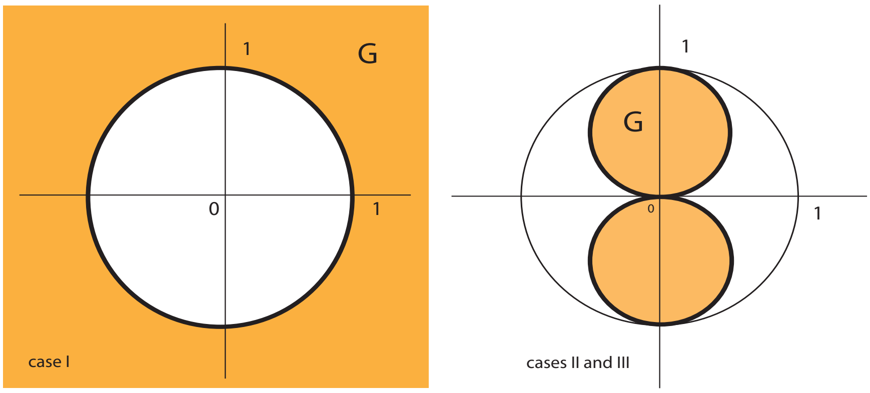

Figure 2: Allowed ranges of G in the complex plane for case I, and cases II and III.

Next, we decompose equation (5.41) into its real and imaginary parts. To do so, decompose G into its real and imaginary parts,

$$G(x,r) = G_r(x,r) + iG_x(x,r) \tag{8.4}$$

for G_r, G_x real functions. The real and imaginary parts of (5.41) are respectively given by,

$$\begin{aligned} \partial_r G_r + \partial_x G_x &= \frac{G_r}{r} \\ \partial_r G_x - \partial_x G_r &= 0 \end{aligned} \tag{8.5}$$

The second equation of (8.5) is solved completely by expressing the components G_r and G_x in terms of the gradient of a single real function. For later convenience, we shall include an extra factor of r in the definition of this function, and denote it by $r\Psi(x,r)$, so that

$$\begin{aligned} G_r &= \partial_r(r\Psi) \\ G_x &= \partial_x(r\Psi) \end{aligned} \tag{8.6}$$

The first equation of (8.5) then becomes a second order partial differential equation on Ψ,

$$\left(\partial_x^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\Psi(x,r) = 0 \tag{8.7}$$

The motivation for including the extra factor of r in the definition of $\Psi(x,r)$ was to assure that the r-part of the above differential equation is of the form of a 2-dimensional Laplace equation in cylindrical coordinates of which r is the radial coordinate.

8.1 Solving by three-dimensional harmonic functions

A simple geometrical interpretation of equation (8.7) is obtained by relating it to the Laplace equation in 3-dimensional Euclidean flat space. All our arguments are local; global regularity conditions will be studied and imposed on the local solutions in a subsequent paper.

JHEP08(2008)028

The solutions of (8.7) are in two-to-one correspondence with harmonic functions in 3-dimensional Euclidean space. To show this, we first note that, if $\Psi(x,r)$ satisfies (8.7), then the real 3-dimensional function $\Phi(x,y,z)$, defined by

$$\Phi(x, r\cos\theta, r\sin\theta) = \Psi(x,r)e^{+i(\theta-\theta_0)} + \Psi(x,r)e^{-i(\theta-\theta_0)} \tag{8.8}$$

satisfies the three-dimensional Laplace equation,

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\Phi(x,y,z) = 0 \tag{8.9}$$

for any relative constant phase θ_0. It is straightforward to check this by casting the 3-dimensional Laplace equation in terms of cyclindrical coordinates r, θ for the directions y, z, using the fact that

$$\frac{\partial^2\Phi(x,y,z)}{\partial\theta^2} = -\Phi(x,y,z) \tag{8.10}$$

and then using (8.7). Conversely, let $\Phi(x,y,z)$ be a solution of the 3-dimensional Laplace equation (8.9). The complex-valued function $\Psi_c(x,r)$, constructed by the Fourier transform,

$$\Psi_c(x,r) = \int_0^{2\pi} d\theta\, e^{i\theta}\, \Phi(x, r\cos\theta, r\sin\theta) \tag{8.11}$$

automatically satisfies (8.7). The real and imaginary parts of $\Psi_c(x,r)$ provide two possible real functions $\Psi(x,r)$ satisfying (8.7). Thus, all solutions to (8.7) may be obtained by this projection method from 3-dimensional harmonic functions.

8.2 Solving by Fourier transform

A more direct method of solving (8.7) involves direct Fourier transformation. Since the equation is invariant under arbitrary translations of the variable x, we use Fourier analysis in x. Since the function $\Psi(x,r)$ is real, we have the following general Fourier representation,

$$\Psi(x,r) = \int_0^\infty \frac{dk}{2\pi}\left(\Psi_k(r)\, e^{-ikx} + \Psi_k(r)^*\, e^{+ikx}\right) \tag{8.12}$$

The individual Fourier modes $\Psi_k(r)$ satisfy the modified Bessel equation, for $r > 0$,

$$\left(-k^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\Psi_k(r) = 0 \tag{8.13}$$

The general solution of this equation for $k \geq 0$ is given by

$$\Psi_k(r) = -\pi\psi_1(k)I_1(kr) + \psi_2(k)K_1(kr) \tag{8.14}$$

where $I_1(kr)$ and $K_1(kr)$ are modified Bessel functions, and $\psi_1(k)$ and $\psi_2(k)$ are arbitrary complex functions of $k \geq 0$. (The extra factor of $-\pi$ has been introduced for later convenience.) The modified Bessel functions admit the following useful integral representations,

$$\begin{aligned} I_1(kr) &= -\frac{1}{\pi}\int_{-1}^{1} \frac{t\,dt}{\sqrt{1-t^2}}\, e^{-tkr} \\ K_1(kr) &= \int_1^\infty \frac{t\,dt}{\sqrt{t^2-1}}\, e^{-tkr} \end{aligned} \tag{8.15}$$

These representations are absolutely convergent for all real $kr > 0$, which is indeed the case here. Next recast the solution $\Psi(x,r)$ in terms of these integral representations,

$$\begin{aligned}\Psi(x,r) = &+\int_0^\infty \frac{dk}{2\pi}\,\psi_1(k)e^{-ikx}\int_{-1}^1 \frac{t\,dt}{\sqrt{1-t^2}}\,e^{-tkr} + \text{c.c.}\\ &+\int_0^\infty \frac{dk}{2\pi}\,\psi_2(k)e^{-ikx}\int_1^\infty \frac{t\,dt}{\sqrt{t^2-1}}\,e^{-tkr} + \text{c.c.}\end{aligned} \tag{8.16}$$

Let us now define the following functions,

$$\begin{aligned}C_1(tr+ix) &\equiv \int_0^\infty \frac{dk}{2\pi}\,\psi_1(k)\,e^{-k(tr+ix)}\\ C_2(tr+ix) &\equiv \int_0^\infty \frac{dk}{2\pi}\,\psi_2(k)\,e^{-k(tr+ix)}\end{aligned} \tag{8.17}$$

and their complex conjugates. Since $\psi_1(k)$ and $\psi_2(k)$ were arbitrary complex functions of k, the functions C_1 and C_2 are arbitrary functions of their argument. The argument is a complex variable $tr + ix$; the functions C_1 and C_2 depend on this variable, but not on its complex conjugate. (The complex conjugated functions $C_1(tr + ix)^*$ and $C_2(tr + ix)^*$ depend on the complex conjugate variable $tr-ix$ but not on $tr+ix$.) Thus, it is appropriate to interpret the functions C_1 and C_2 as *holomorphic functions of their argument.* The functions entering the real integrals are thus harmonic functions, but not of w, but instead of $th + i\tilde{h}$. Then the solution $\Psi(x,r)$ may be expressed as follows,

$$\begin{aligned}\Psi(x,r) = &\int_{-1}^1 \frac{t\,dt}{\sqrt{1-t^2}}\,(C_1(tr+ix) + C_1(tr+ix)^*)\\ &+\int_1^\infty \frac{t\,dt}{\sqrt{t^2-1}}\,(C_2(tr+ix) + C_2(tr+ix)^*)\end{aligned} \tag{8.18}$$

It is straightforward to show directly that this expression is a solution to the original differential equation (8.7) for any holomorphic functions C_1 and C_2, by using integration by parts. But, using the steps we have taken, we now know that this is the most general solution. One may also directly work out the expression for G,

$$\begin{aligned}G(x,r) = &\,r\int_{-1}^1 \frac{dt}{\sqrt{1-t^2}}\,\big((1-t)C_1'(tr+ix) + (1+t)C_1'(tr+ix)^*\big)\\ &+r\int_1^\infty \frac{dt}{\sqrt{t^2-1}}\,\big((1-t)C_2'(tr+ix) + (1+t)C_2'(tr+ix)^*\big)\end{aligned} \tag{8.19}$$

This result gives the complete local solution to the BPS equations in exact form. The above expression is indeterminate in the limit $r \to 0$ due to the divergence of the integrals and the overall factor of r. It will therefore be useful to have an asymptotic form for $G(r,x)$ near $r = 0$. First we expand Ψ in a series about $r = 0$ using the expression (8.14)

$$\Psi_k(r) \approx \psi_2(k)\left(\frac{1}{r} + \frac{1}{2}r\ln r\right) + \mathcal{O}(r) \tag{8.20}$$

Computing first $\Psi(x,r)$ and then $G(x,r)$ using (8.6) we obtain

$$G(x,r) = \int_0^\infty \frac{dk}{2\pi}\,\Big(k\,\psi_2(k)e^{-ikx} + c.c.\Big) + \mathcal{O}(r) \tag{8.21}$$

which is just the statement that $G(x,0)$ is an arbitrary function of x.

8.3 Equations for Ψ and G in general coordinates

From equation (8.6), we deduce that

$$G = G_r + iG_x = 2\partial_{\bar u}(h\Psi) \qquad\qquad u = r + ix = h + i\tilde h \tag{8.22}$$

This expression is cast in the form of the special coordinates u adapted to the arbitrary harmonic function h. We can, however, also express G in terms of Ψ with general coordinates w, by changing conformal coordinates from u to w. We use, $\partial_w u = \partial_w(h + i\tilde h) = 2\partial_w h$, and its inverse, to give the following form for G in arbitrary conformal coordinates w,

$$G = \frac{\partial_{\bar w}(h\Psi)}{\partial_{\bar w} h} \tag{8.23}$$

The first order differential equation for G of (5.41) may be recast in terms of a second order differential equation for $h\Psi$, for a general coordinate system $w, \bar w$, by eliminating G in terms of $h\Psi$. This equation is given by,

$$2\partial_w\partial_{\bar w}\Psi + \partial_w\Psi\partial_{\bar w}\ln h + \partial_{\bar w}\Psi\partial_w \ln h + 2\Psi\partial_w\partial_{\bar w}\ln h = 0 \tag{8.24}$$

valid for arbitrary conformal coordinates w and $\bar w$. The general solution to this equation is simply given by changing variables in the result for Ψ expressed in terms of coordinates x, y in (8.18), and we find, in general coordinates $w, \bar w$,

$$\begin{aligned}\Psi(\tilde h, h) \;=\;& \int_{-1}^{1} \frac{t\,dt}{\sqrt{1-t^2}} \Big(C_1(th + i\tilde h) + C_1(th + i\tilde h)^* \Big) \\ &+ \int_{1}^{\infty} \frac{t\,dt}{\sqrt{t^2-1}} \Big(C_2(th + iy\tilde h) + C_2(th + iy\tilde h)^* \Big)\end{aligned} \tag{8.25}$$

Similarly, G in general coordinates takes the form,

$$\begin{aligned}G(\tilde h, h) \;=\;& h\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \Big((1-t)C_1'(th + i\tilde h) + (1+t)C_1'(th + i\tilde h)^* \Big) \\ &+h \int_{1}^{\infty} \frac{dt}{\sqrt{t^2-1}} \Big((1-t)C_2'(th + i\tilde h) + (1+t)C_2'(th + i\tilde h)^* \Big)\end{aligned} \tag{8.26}$$

In both equations, it is understood that h and $\tilde h$ are to be expressed as functions of $w, \bar w$.

8.4 The inverse problem of determining C_1, C_2 from G

In the case of solutions G such that

$$G_\partial(\tilde h) = \lim_{h\to 0}\left(\frac{G}{h}\right) \tag{8.27}$$

is finite, there is a simple method of recovering the functions $C_{1,2}$ from G. Assuming that G_∂ is a finite function of $\tilde h$, we take the corresponding limit in the integral of (8.26). Both integrals involving C_2 on the second line of (8.26) diverge in the limit $h \to 0$. Canceling the leading linear divergence requires $C_2'(\tilde h) = C_2'(\tilde h)^*$ for all $\tilde h$. Canceling also the remaining logarithmic divergence requires $C_2'(\tilde h) = 0$ for all $\tilde h$. Since $C_2(th + i\tilde h)$ is a holomorphic function of $th+i\tilde{h}$ which vanishes on the line $h=0$, it must be identically zero, $C_2(th+i\tilde{h})=0$. The remaining t-integrals multiplying $C_1'(\tilde{h})$ and $C_1'(\tilde{h})^*$ may be evaluated and equal π. We thus obtain

$$C_1'(i\tilde{h}) + C_1'(i\tilde{h})^* = \frac{G_\partial(\tilde{h})}{\pi} \tag{8.28}$$

This is enough information to determine the real part of the C_1'. In addition, it shows that G is real on the boundary defined by $h=0$ and the imaginary part of the C_1' only affects the value of G away from the boundary defined by $h=0$.

8.5 The functions Ψ, C_1, C_2 for the $AdS_7 \times S^4$ case

We now proceed to evaluate the functions Ψ, C_1 and C_2 for the $AdS_7 \times S^4$ case. First we quote expressions for G, h, and $\tilde{h}$,

$$\begin{aligned} G(w,\bar{w}) &= -i\frac{\mathrm{sh}(w-\bar{w})}{\mathrm{sh}(2\bar{w})} \\ r = h(w,\bar{w}) &= -i(\mathrm{ch}(2w) - \mathrm{ch}(2\bar{w})) \\ x = \tilde{h}(w,\bar{w}) &= -(\mathrm{ch}(2w) + \mathrm{ch}(2\bar{w})) \end{aligned} \tag{8.29}$$

The expression for $h\Psi$ is readily computed and we find,

$$h\Psi(\tilde{h},h) = -2\mathrm{ch}(w-\bar{w}) \tag{8.30}$$

To obtain $\Psi(\tilde{h},h)$ as a function of $\tilde{h}$ and h is surprisingly complicated, as Ψ is found to satisfy a 4-th degree polynomial equation,

$$\frac{\tilde{h}^2}{h^2\Psi^2} + \frac{h^2}{h^2\Psi^2 - 4} = 1 \tag{8.31}$$

In other words, the equipotential lines for $h\Psi(\tilde{h},h)$ are ellipses, when $|h\Psi(\tilde{h},h)| \geq 2$, and hyperbolas, when $|h\Psi(\tilde{h},h)| \leq 2$. It would be difficult to carry out the Fourier integrals needed to derive the functions C_1 and C_2. Fortunately, we may use the more indirect methods discussed above to extract C_1 and C_2 from the limiting behavior as $h \to 0$. As a result, we begin by first calculating the ratio G/h. The harmonic function h vanishes whenever $w - \bar{w} = in\pi$ for any integer n, so that $w = s + in\pi/2$, for any real s. For simplicity, consider the case $n=0$, the other cases being completely analogous. For these values, we have $\tilde{h} = -2\mathrm{ch}(2s)$, and

$$G_\partial = \frac{1}{2\,\mathrm{sh}(2s)^2} = -\frac{2}{(i\tilde{h})^2 + 4} \tag{8.32}$$

The limit $h \to 0$ of G/h is a finite function G_∂ of $\tilde{h}$, and we conclude that we must have $C_2 = 0$. The result further determines the real part of the holomorphic function $C_1'(th+i\tilde{h})$ on the boundary $h=0$. Assuming that the imaginary part of this function vanishes for $h=0$, we find that

$$C_1'(v) = -\frac{1}{\pi}\frac{1}{v^2+4} = \frac{i}{4\pi}\left(\frac{1}{v-2i} - \frac{1}{v+2i}\right) \tag{8.33}$$

for all complex $v = th + i\tilde{h}$. In turn, substituting this expression and $C_2 = 0$ into (8.26), and performing the t-integrations yields back expression (8.29) for G in terms of $w, \bar{w}$, as we shall show explicitly in section 8.7 below.

8.6 The function Ψ for the $AdS_4 \times S^7$ case

Next, we proceed to evaluating the functions Ψ, C_1 and C_2 for the $AdS_4 \times S^7$ case. First we quote expressions for G, h, and $\tilde{h}$,

$$\begin{aligned} G(w,\bar{w}) &= i\frac{\mathrm{ch}(w+\bar{w})}{\mathrm{ch}(2\bar{w})} \\ r = h(w,\bar{w}) &= 4i(\mathrm{sh}(2w) - \mathrm{sh}(2\bar{w})) \\ x = \tilde{h}(w,\bar{w}) &= 4(\mathrm{sh}(2w) + \mathrm{sh}(2\bar{w})) \end{aligned} \tag{8.34}$$

The expression for $h\Psi$ is readily computed and we find,

$$h\Psi(\tilde{h},h) = 8\,\mathrm{sh}(w+\bar{w}) \tag{8.35}$$

To obtain $\Psi(\tilde{h},h)$ as a function of $\tilde{h}$ and h is surprisingly complicated, as Ψ is found to satisfy a 4-th degree polynomial equation,

$$\frac{\tilde{h}^2}{h^2\Psi^2} + \frac{h^2}{h^2\Psi^2+64} = 1 \tag{8.36}$$

The equipotential lines for $h\Psi(\tilde{h},h)$ are always ellipses.

8.7 Simple poles in C_1' and C_2'

To derive the functions C_1 and C_2 of (8.18) for the case $AdS_4 \times S^7$, it would be difficult to carry out the Fourier integrals on Ψ, and its is also not possible to use the methods of section 8.4, since the quantity G/h does not have a finite limit as $h \to 0$. Having the result for the $AdS_7 \times S^4$ case in terms of simple poles for C_1, it is natural to see a result for $AdS_4 \times S^7$ also in terms of simple poles for C_1 and C_2. This will allow us to check, by direct calculation, that both cases correspond to simple poles for C_1' and C_2'.

The key ingredients are the following integrals, for $z \in \mathbf{C}$,

$$\begin{aligned} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}\,\frac{1}{t+z} &= \frac{\pi}{\sqrt{z^2-1}} \\ \int_{1}^{\infty} \frac{dt}{\sqrt{t^2-1}}\,\frac{1}{t+z} &= \frac{1}{\sqrt{z^2-1}}\,\ln\Big(z+\sqrt{z^2-1}\Big) \end{aligned} \tag{8.37}$$

Here, the branches have been defined so that for $z \in \mathbf{R}$ and $z > 1$ both integrals are real and positive, with a real and positive branch is chosen for the square rots and for the logarithm. Analytic continuations $z \to -z$ must be carried out with care, and we find,

$$\begin{aligned} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}}\,\frac{1}{t-z} &= -\frac{\pi}{\sqrt{z^2-1}} \\ \int_{1}^{\infty} \frac{dt}{\sqrt{t^2-1}}\,\frac{1}{t-z} &= -\frac{1}{\sqrt{z^2-1}}\,\ln\Big(-z-\sqrt{z^2-1}\Big) \end{aligned} \tag{8.38}$$

In particular, by putting together the following combinations,

$$\int_1^\infty \frac{dt}{\sqrt{t^2-1}} \left(\frac{1}{t+z} + \frac{1}{t-z} \right) = \frac{\pi}{\sqrt{1-z^2}} \tag{8.39}$$

Using these integrals we may now check that C_1' of (8.33) together with $C_2' = 0$ indeed yields G as given by (8.29). We shall also evaluate the contribution to G from a simple pole in C_2'. Both calculations result from considering either C_1' or C_2' to be of the form,

$$C'(v) = \frac{a}{v+b} \tag{8.40}$$

for $a, b \in \mathbf{C}$. The integrands for both calculations may be simplified as follows,

$$r(1-t)C'(tr+ix) + r(1+t)C'(tr+ix)^* = -(a-\bar{a}) + a\frac{z+1}{t+z} - \bar{a}\frac{\bar{z}-1}{t+\bar{z}} \tag{8.41}$$

where we have used the abbreviation

$$z = (ix+b)/r \tag{8.42}$$

8.7.1 Contributions from poles in C_1'

The contribution $G_1(x,r)$ to $G(x,r)$ of a simple pole (8.40) in C_1' is given by,

$$\begin{aligned} G_1(x,r) &= -\pi(a-\bar{a}) + \pi a \sqrt{\frac{z+1}{z-1}} - \pi\bar{a}\sqrt{\frac{\bar{z}-1}{\bar{z}+1}} \\ &= -\pi(a-\bar{a}) + \pi a \sqrt{\frac{r+ix+b}{-r+ix+b}} - \pi\bar{a}\sqrt{\frac{-r-ix+\bar{b}}{r-ix+\bar{b}}} \end{aligned} \tag{8.43}$$

For purely imaginary b, the two square roots are equal to one another. To recover (8.33), we add the contribution from $b=-2i$ and the opposite of that from $b=2i$, with the same value of a cancels the $-\pi(a-\bar{a})$ term, and gives,

$$G(x,r) = \pi(a-\bar{a}) \left(\sqrt{\frac{r+ix-2i}{-r+ix-2i}} - \sqrt{\frac{r+ix+2i}{-r+ix+2i}} \right) \tag{8.44}$$

Using now the combinations

$$\begin{aligned} r+ix &= -2i\mathrm{ch}(2w) \\ r-ix &= +2i\mathrm{ch}(2\bar{w}) \end{aligned} \tag{8.45}$$

gives

$$G(x,r) = \pi(a-\bar{a}) \left(\frac{\mathrm{ch}w}{\mathrm{ch}\bar{w}} - \frac{\mathrm{sh}w}{\mathrm{sh}\bar{w}} \right) = -2\pi(a-\bar{a}) \frac{\mathrm{sh}(w-\bar{w})}{\mathrm{sh}(2\bar{w})} \tag{8.46}$$

Clearly, we must choose $a = i/(4\pi)$ to recover (8.29), and this value agrees with the one found in (8.33).

8.7.2 Contributions from poles in C_2'

The contribution $G_2(x,r)$ to $G(x,r)$ of a simple pole (8.40) in C_2', is given by

$$\begin{aligned} G_2(x,r) &= -(a-\bar a)\ln\Lambda + a\sqrt{\frac{z+1}{z-1}}\ln\left(z+\sqrt{z^2-1}\right) \\ &\quad -\bar a\sqrt{\frac{\bar z-1}{\bar z+1}}\ln\left(\bar z+\sqrt{\bar z^2-1}\right) \end{aligned} \tag{8.47}$$

where we have again used the definition (8.42) for z, and Λ is a constant cutoff for the divergent integral over t. We now add the contributions for (a,b) and the opposite of $(\bar a,-\bar b)$, which is a combination chosen so that the logarithms will only contribute through their discontinuities, namely though $\ln(z)-\ln(-z)=-i\pi$. The result for C_2' is given by,

$$C_2'(v) = \frac{a}{v+b} - \frac{\bar a}{v-\bar b} \tag{8.48}$$

and the resulting contribution to G is given by

$$G(x,r) = -i\pi a\sqrt{\frac{r+ix+b}{-r+ix+b}} - i\pi\bar a\sqrt{\frac{-r-ix+\bar b}{r-ix+\bar b}} \tag{8.49}$$

Using now the combinations

$$\begin{aligned} r+ix &= +8i\mathrm{sh}(2w) \\ r-ix &= -8i\mathrm{sh}(2\bar w) \end{aligned} \tag{8.50}$$

and the choice $b=-8$, we find,

$$G(x,r) = -i\pi a\sqrt{\frac{i\,\mathrm{sh}(2w)-1}{i\,\mathrm{sh}(2\bar w)-1}} - i\pi\bar a\sqrt{\frac{i\,\mathrm{sh}(2w)+1}{i\,\mathrm{sh}(2\bar w)+1}} \tag{8.51}$$

Using now the identities,

$$i\,\mathrm{sh}(2w)\pm 1 = \pm 2\mathrm{ch}\left(w\pm i\pi/4\right)^2 \tag{8.52}$$

and their complex conjugate relations, we readily show that

$$G(x,r) = -i\pi a\frac{\mathrm{ch}(w-i\pi/4)}{\mathrm{ch}(\bar w-i\pi/4)} - i\pi\bar a\frac{\mathrm{ch}(w+i\pi/4)}{\mathrm{ch}(\bar w+i\pi/4)} \tag{8.53}$$

To recover the expression for G of (8.34), it suffices to take $C_1'=0$, a real and equal to $a=-1/(4\pi)$, and to use the following identities,

$$\begin{aligned} \mathrm{ch}(2\bar w) &= 2\mathrm{ch}(\bar w+i\pi/4)\mathrm{ch}(\bar w-i\pi/4) \\ \mathrm{ch}(w+\bar w) &= \mathrm{ch}(w-i\pi/4)\mathrm{ch}(\bar w+i\pi/4)+\mathrm{ch}(w+i\pi/4)\mathrm{ch}(\bar w-i\pi/4) \end{aligned} \tag{8.54}$$

This concludes our calculation of C_1' and C_2' for the case $AdS_4\times S^7$.

9. Bianchi identities and field equations

We have now completely solved the BPS equations for cases I, II, and III, by reducing the problem to a set of linear equations. We shall now show that the Bianchi identities and the field equations are automatically satisfied, as soon as the BPS equations hold.

We shall present explicitly only the case III, for which $c_1 = c_2$. The fact that cases I and II are related to case III by analytic continuation then automatically guarantees that the Bianchi identities and field equations follow from the BPS equations also for cases I and II.

9.1 Bianchi identities

The reduced Bianchi identities (2.11) may be expressed as,

$$\partial_w b_i = -\rho f_i^3 g_{zi} \qquad\qquad i = 1,2,3 \tag{9.1}$$

where the b_i are the real gauge potentials. Using equation (3.7) for the metric factors and equation (5.1) for the fluxes in terms of α, β and ψ, integrability of the Bianchi identities is equivalent to conservation of the following currents

$$\begin{aligned}
\partial_w b_1 \sim j^1_w &= \rho(\bar\alpha\alpha + \bar\beta\beta)^3(\alpha^2+\beta^2)\psi \\
\partial_w b_2 \sim j^2_w &= -\rho(\bar\alpha\alpha - \bar\beta\beta)^3(\alpha^2-\beta^2)\psi \\
\partial_w b_3 \sim j^3_w &= \left(\frac{\bar\beta}{\beta} - \frac{\bar\alpha}{\alpha}\right)^3 \left[-3\rho\alpha^2\beta^2 + \rho(\alpha^4-\beta^4)\alpha^2\beta^2\psi\right]
\end{aligned} \tag{9.2}$$

where the $\sim$ sign stands for equality up to an overall constant factor. Current conservation is expressed here as the closure of differential forms, and takes the form,

$$\partial_{\bar w} j^i_w - \partial_w j^i_{\bar w} = 0 \qquad\qquad i = 1,2,3 \tag{9.3}$$

Instead of working with j^1_w and j^2_w, it will be convenient to work with $j^\pm_w$ defined as

$$j^\pm_w = \frac{1}{2}(j^1_w \pm j^2_2) \tag{9.4}$$

To confirm that the BPS equations imply the Bianchi identities, we first re-write the currents in terms of G and $\bar G$ and then use the remaining BPS equations (8.6) and (8.7) to show that the Bianchi identities are automatically satisfied when the BPS equations are.

As a first step we express the Bianchi identities in terms of φ and $\tilde\rho$ using equations (5.8) to eliminate α and β, equation (5.20) to eliminate ρ, and equation (5.17) as well as the fact $\partial_w\bar\sigma = 2\bar\kappa\,\mathrm{sh}(2\bar\varphi)\partial_w\bar\varphi$ to eliminate ψ,

$$\begin{aligned}
j^+_w &= \frac{\sqrt{2}\partial_w\bar\varphi}{\tilde\rho^{3/2}\,\mathrm{sh}(2\bar\varphi)\mathrm{sh}(\varphi+\bar\varphi)}\left(2\mathrm{ch}(\varphi+\bar\varphi)\mathrm{ch}(2\bar\varphi) - \mathrm{ch}(\varphi-\bar\varphi)\right) \\
j^-_w &= -\frac{\partial_w\bar\varphi}{\sqrt{2}\tilde\rho^{3/2}\,|\mathrm{sh}(2\varphi)|\mathrm{sh}(\varphi+\bar\varphi)}\left(\mathrm{sh}(2\bar\varphi)\mathrm{ch}(2\varphi) + 3\mathrm{sh}(2\varphi)\mathrm{ch}(2\bar\varphi)\right) \\
j^3_w &= \frac{\mathrm{sh}(\varphi-\bar\varphi)^3}{\mathrm{sh}(2\bar\varphi)|\mathrm{sh}(2\varphi)|}\left(-12\kappa + 8\sqrt{2}\frac{\partial_w\bar\varphi}{\tilde\rho^{3/2}}\,\frac{\mathrm{sh}\varphi(\mathrm{ch}\bar\varphi)^3 - \mathrm{ch}\varphi(\mathrm{sh}\bar\varphi)^3}{|\mathrm{sh}(2\varphi)|^2\,\mathrm{sh}(\varphi+\bar\varphi)}\right)
\end{aligned} \tag{9.5}$$

Next we recast the conserved currents in terms of the variables h, ϑ and μ. This is done by using the definition of $\hat{\rho}$ (5.31) and the fact $\hat{\rho}^{-3/2} = h$ as well as the definitions for ϑ and μ (5.24) and the identities (5.25) and (5.26). Finally, we may use the definition of G (5.40) to write the conserved currents in terms of G and $\bar{G}$ as follows

$$\begin{aligned} j_w^+ &= 2ih\,\left(\bar{G}(G-3\bar{G}+4G\bar{G}^2)\partial_w G + G(G+\bar{G})\partial_w \bar{G}\right) \\ &\quad \times \left((G-\bar{G})^2 - 4G^3\bar{G}\right)(G+\bar{G})^{-1}W^{-4} \\ j_w^- &= 2h\,G\left(\bar{G}(G-3\bar{G}+4G\bar{G}^2)\partial_w G + G(G+\bar{G})\partial_w \bar{G}\right) \\ &\quad \times \left(-2G\bar{G}+3\bar{G}^2-G^2+4G^2\bar{G}^2\right)(G+\bar{G})^{-1}W^{-4} \\ j_w^3 &= 3\partial_w h\,\frac{W^2}{G(1-G\bar{G})} - 2h\,\frac{(1+G^2)}{G(G+\bar{G})(1-G\bar{G})^2} \\ &\quad \times \left(\bar{G}(G-3\bar{G}+4G\bar{G}^2)\partial_w G + G(G+\bar{G})\partial_w \bar{G}\right) \end{aligned} \tag{9.6}$$

Integrability may now be checked as follows. First we choose conformal coordinates such that $h = r$ as in section (8). Next we eliminate G and $\bar{G}$ in terms of Ψ using (8.6). Next we compute $\partial_{\bar{w}} j_w^i - \partial_w j_{\bar{w}}^i$ where $i = +,-,3$. Upon using the fact h is harmonic and the differential equation for Ψ (8.7) to eliminate the terms with second order derivatives and higher in r the resulting expressions will automatically vanish. This shows that the BPS equations imply the Bianchi identities are automatically satisfied. Finally, we give the expressions for the $\partial_w b_i$ with the correct normalization factors

$$\begin{aligned} \partial_w b_1 &= -2(j_w^+ + j_w^-) \\ \partial_w b_2 &= 2(j_w^+ - j_w^-) \\ \partial_w b_3 &= -\frac{1}{8}\,j_w^3 \end{aligned} \tag{9.7}$$

9.2 Field equations for the 4-form F

The field equation for the 4-form F is given by (2.3)[5]

$$d * F + \frac{1}{2} F \wedge F = 0 \tag{9.8}$$

Writing out in components we have

$$\begin{aligned} 0 &= \partial_{\bar{w}}\partial_w b_1 + \frac{1}{2}\left(\partial_{\bar{w}} b_1 \partial_w \ln\left(\frac{f_2 f_3}{f_1}\right)^3 + c.c.\right) + \frac{i}{4}\left(\frac{f_1}{f_2 f_3}\right)^3 (\partial_w b_2 \partial_{\bar{w}} b_3 - c.c.) \\ 0 &= \partial_{\bar{w}}\partial_w b_2 + \frac{1}{2}\left(\partial_{\bar{w}} b_2 \partial_w \ln\left(\frac{f_1 f_3}{f_2}\right)^3 + c.c.\right) + \frac{i}{4}\left(\frac{f_2}{f_1 f_3}\right)^3 (\partial_w b_1 \partial_{\bar{w}} b_3 - c.c.) \\ 0 &= \partial_{\bar{w}}\partial_w b_3 + \frac{1}{2}\left(\partial_{\bar{w}} b_3 \partial_w \ln\left(\frac{f_1 f_2}{f_3}\right)^3 + c.c.\right) - \frac{i}{4}\left(\frac{f_3}{f_1 f_2}\right)^3 (\partial_w b_1 \partial_{\bar{w}} b_2 - c.c.) \end{aligned} \tag{9.9}$$

---

[5] Our convention for the Hodge dual is given as follows. Let $F_{(p)}$ be a p-form $F_{(p)} = \frac{1}{p!} F_{a_1 a_2 \dots a_p} e^{a_1 a_2 \dots a_p}$. The Hodge dual is defined by $*F_{(p)} = \frac{1}{p!(11-p)!}\epsilon_{a_1 a_2 \dots a_p b_{p+1} b_{p+2} \dots b_{11}} F^{a_1 a_2 \dots a_p} e^{b_{p+1} b_{p+2} \dots b_{11}}$, and $\varepsilon_{a_1 a_2 \dots a_{11}}$ is the anti-symmetric tensor with $\varepsilon_{0123\dots 9\natural} = +1$. In particular, we have the following results needed in the calculations of the present paper, $*e^{012a} = -\epsilon^a{}_b e^{345678b}$, $*e^{345a} = -\epsilon^a{}_b e^{012678b}$ and $*e^{678a} = +\epsilon^a{}_b e^{012678b}$.

where we have made use of the relation (2.11) to write the equations in terms of the b_i.

Using the BPS equations, the 4-form equations (9.9) may be shown to be satisfied automatically. This is done as follows. First we express the equations in terms of G, $\bar{G}$ and h using (9.7) for the fluxes. Next, we choose a conformal gauge such that $h = r$ as in section (8). Next, we re-express G in terms of Ψ using (8.6). Finally we use the differential equation (8.7) for Ψ to eliminate terms with second order derivatives and higher in r. The resulting equations will then be satisfied automatically. The resulting calculations are lengthy and will not be repeated here; they were confirmed using MATHEMATICA.

9.3 Einstein's equations

Einstein's equations are given by (2.2)

$$R_{MN} - \frac{1}{12}F_{MPQR}F_N{}^{PQR} + \frac{1}{144}g_{MN}F_{PQRS}F^{PQRS} = 0 \tag{9.10}$$

To obtain Einstein's equations, we will need to calculate the Ricci tensor R_{MN} this is most easily done by first computing the curvature two-form $\Omega^A{}_B$ and then the Ricci tensor using

$$\begin{aligned}\Omega^A{}_B &= d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \\ i_{e^A}\Omega^A{}_B &= R_{BD}e^D\end{aligned} \tag{9.11}$$

The expressions for the curvature two-forms on the symmetric spaces AdS_3, S_2^3 and S_3^3 as well as the two-dimensional base space Σ are given by

$$\begin{aligned}\hat{\Omega}^m{}_n &= -\hat{e}^m \wedge \hat{e}_n & \qquad \hat{\Omega}^{i_1}{}_{j_1} &= +\hat{e}^{i_1} \wedge \hat{e}_{j_1} \\ \hat{\Omega}^{i_2}{}_{j_2} &= +\hat{e}^{i_2} \wedge \hat{e}_{j_2} & \qquad \Omega^a{}_b &= R^{(2)}e^a \wedge e^b\end{aligned} \tag{9.12}$$

The curvature two-form form can then computed using the above equations and the expressions for the spin-connection (2.17). The *block diagonal entries* are given as follows,

$$\begin{aligned}\Omega^m{}_n &= \left(-\frac{1}{f_1^2} - |D_a \ln f_1|^2\right) e^m \wedge e_n \\ \Omega^{i_1}{}_{j_1} &= \left(+\frac{1}{f_2^2} - |D_a \ln f_2|^2\right) e^{i_1} \wedge e_{j_1} \\ \Omega^{i_2}{}_{j_2} &= \left(+\frac{1}{f_3^2} - |D_a \ln f_3|^2\right) e^{i_2} \wedge e_{j_2} \\ \Omega^a{}_b &= R^{(2)}e^a \wedge e^b\end{aligned} \tag{9.13}$$

where we use the notation $|D_a f|^2 \equiv D^a f D_a f$. The *block off-diagonal entries* between two of the factor spaces AdS_3, S_2^3 and S_3^3 are given by,

$$\begin{aligned}\Omega^m{}_{i_1} &= -(D^a \ln f_2)(D_a \ln f_1)e^m \wedge e_{i_1} \\ \Omega^m{}_{i_2} &= -(D^a \ln f_3)(D_a \ln f_1)e^m \wedge e_{i_2} \\ \Omega^{i_1}{}_{i_2} &= -(D^a \ln f_2)(D_a \ln f_3)e^{i_1} \wedge e_{i_2}\end{aligned} \tag{9.14}$$

The *block off-diagonal entries* between one of the factor spaces AdS_3, S_2^3 and S_3^3 and the surface Σ are given by,

$$\begin{aligned}
\Omega^m{}_a &= \frac{1}{f_1}(D_b D_a f_1) + \epsilon^c{}_a \frac{D_c f_1}{f_1} e^m \wedge \hat\omega \\
\Omega^{i_1}{}_a &= \frac{1}{f_2}(D_b D_a f_2) e^b \wedge e^{i_1} + \epsilon^c{}_a \frac{D_c f_2}{f_2} e^{i_1} \wedge \hat\omega \\
\Omega^{i_2}{}_a &= \frac{1}{f_3}(D_b D_a f_3) e^b \wedge e^{i_2} + \epsilon^c{}_a \frac{D_c f_3}{f_3} e^{i_2} \wedge \hat\omega
\end{aligned} \tag{9.15}$$

The resulting components of the Ricci tensor (in frame index convention) are given by

$$\begin{aligned}
R_{mn} &= \eta_{mn}\left(-\frac{2}{f_1^2} - 2|D_a \ln f_1|^2 - 3(D^a \ln f_1)(D_a \ln f_2 f_3) - \frac{D^a D_a f_1}{f_1} - \hat\omega_a (D^b \ln f_1)\epsilon_b{}^a\right) \\
R_{i_1 j_1} &= \delta_{i_1 j_1}\left(+\frac{2}{f_2^2} - 2|D_a \ln f_2|^2 - 3(D^a \ln f_2)(D_a \ln f_1 f_3) - \frac{D^a D_a f_2}{f_2} - \hat\omega_a (D^b \ln f_2)\epsilon_b{}^a\right) \\
R_{i_2 j_2} &= \delta_{i_2 j_2}\left(+\frac{2}{f_3^2} - 2|D_a \ln f_3|^2 - 3(D^a \ln f_3)(D_a \ln f_1 f_2) - \frac{D^a D_a f_3}{f_3} - \hat\omega_a (D^b \ln f_3)\epsilon_b{}^a\right) \\
R_{ab} &= -3\frac{D_b D_a f_1}{f_1} - 3\frac{D_b D_a f_2}{f_2} - 3\frac{D_b D_a f_3}{f_3} - 3\,\hat\omega_b (D_c \ln f_1)\epsilon^c{}_a \\
&\quad -3\,\hat\omega_b (D_c \ln f_2)\epsilon^c{}_a - 3\,\hat\omega_b (D_c \ln f_3)\epsilon^c{}_a + R^{(2)}\delta_{ab}
\end{aligned} \tag{9.16}$$

while all other components vanish. $R^{(2)}$ is the two-dimensional curvature of Σ, and may be computed using the conventions in § 2.6. It is given by

$$R^{(2)} = -\frac{1}{\rho^2}\partial_w \partial_{\bar w} \ln \rho \tag{9.17}$$

The contributions from the 4-form F are as follows,

$$-\frac{1}{12}F_{MPQR}F_N{}^{PQR} + \frac{1}{144} g_{MN} F_{PQRS}F^{PQRS} = \\
\begin{cases}
\eta_{mn}\left(+\frac{1}{3}g_{1a}^2 + \frac{1}{6}g_{2a}^2 + \frac{1}{6}g_{3a}^2\right) & (M,N) = (m,n) \\
\delta_{i_1 j_1}\left(-\frac{1}{6}g_{1a}^2 - \frac{1}{3}g_{2a}^2 + \frac{1}{6}g_{3a}^2\right) & (M,N) = (i_1,j_1) \\
\delta_{i_2 j_2}\left(-\frac{1}{6}g_{1a}^2 + \frac{1}{6}g_{2a}^2 - \frac{1}{3}g_{3a}^2\right) & (M,N) = (i_2,j_2) \\
\frac{1}{2}g_{1a}g_{1b} - \frac{1}{2}g_{2a}g_{2b} - \frac{1}{2}g_{3a}g_{3b} & (M,N) = (a,b) \\
\quad +\delta_{ab}\left(-\frac{1}{6}g_{1a}^2 + \frac{1}{6}g_{2a}^2 + \frac{1}{6}g_{3a}^2\right) & \\
0 & \text{otherwise}
\end{cases} \tag{9.18}$$

The equations along the symmetric spaces AdS_3, S_1^3, and S_2^3 are

$$\begin{aligned}
0 &= -\frac{2\rho^2}{f_1^2} - 2|\partial_w \ln f_1|^2 - \frac{3}{2}\left((\partial_w \ln f_1)(\partial_{\bar w} \ln f_2 f_3) + c.c.\right) - \frac{\partial_w \partial_{\bar w} f_1}{f_1} \\
&\quad + \frac{1}{3}\frac{|\partial_w b_1|^2}{f_1^6} + \frac{1}{6}\frac{|\partial_w b_2|^2}{f_2^6} + \frac{1}{6}\frac{|\partial_w b_3|^2}{f_3^6} \\
0 &= +\frac{2\rho^2}{f_2^2} - 2|\partial_w \ln f_2|^2 - \frac{3}{2}\left((\partial_w \ln f_2)(\partial_{\bar w} \ln f_1 f_3) + c.c.\right) - \frac{\partial_w \partial_{\bar w} f_2}{f_2} \\
&\quad - \frac{1}{6}\frac{|\partial_w b_1|^2}{f_1^6} - \frac{1}{3}\frac{|\partial_w b_2|^2}{f_2^6} + \frac{1}{6}\frac{|\partial_w b_3|^2}{f_3^6}
\end{aligned}$$

$$0 = +\frac{2\rho^2}{f_3^2} - 2|\partial_w \ln f_3|^2 - \frac{3}{2}\left((\partial_w \ln f_3)(\partial_{\bar w} \ln f_1 f_2) + c.c.\right) - \frac{\partial_w \partial_{\bar w} f_3}{f_3}$$
$$-\frac{1}{6}\frac{|\partial_w b_1|^2}{f_1^6} + \frac{1}{6}\frac{|\partial_w b_2|^2}{f_2^6} - \frac{1}{3}\frac{|\partial_w b_3|^2}{f_3^6} \tag{9.19}$$

With respect to the frame rotation group SO(2) of Σ, all three equations are of weight $(0,0)$. The equations along Σ contain both a weight $(0,0)$ part and a weight $(2,0)$ part, and it will be useful to separate them. They are correspondingly

$$0 = -3\sum_{i=1}^{3} \frac{\partial_w \partial_{\bar w} f_i}{f_i} + 2\rho^2 R^{(2)} + \frac{1}{6}\left(\frac{|\partial_w b_1|}{f_1^6} - \frac{|\partial_w b_2|^2}{f_2^6} - \frac{|\partial_w b_3|^2}{f_3^6}\right)$$
$$0 = -3\sum_{i=1}^{3} \left(\frac{\partial_w^2 f_i}{f_i} - 2\left(\partial_w \ln \rho\right)\left(\partial_w \ln f_i\right)\right) + \frac{1}{2}\left(\frac{|\partial_w b_1|}{f_1^6} - \frac{|\partial_w b_2|^2}{f_2^6} - \frac{|\partial_w b_3|^2}{f_3^6}\right) \tag{9.20}$$

Using the BPS equations, the Einstein equations (9.19) and (9.20) may be shown to hold automatically in a manner similar to the arguments given for the field equations of the 4-form F. First we express the reduced Einstein equations in terms of G, $\bar G$ and h using (5.46), (5.45), and (5.44) for the metric factors. We choose conformal coordinates such that $h = r$ and re-express G in terms of Ψ using (8.6). Finally we use the differential equation for Ψ (8.7) to eliminate terms with second order derivatives and higher in r. Again, these calculations were checked using MATHEMATICA.

A. Clifford algebra basis adapted to the ansatz

The Clifford algebra of 11-dimensional Γ-matrices is defined by

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \tag{A.1}$$

where A, B are 11-dimensional Lorentz frame indices, $A, B = 0, 1, \ldots, 9, \natural = 10$, and η^{AB} is the Minkowski metric $\eta^{AB} = \mathrm{diag}(- + \cdots + +)$. A choice of Γ-matrices which is well adapted to the product structure $AdS_3 \times S_2^3 \times S_3^3 \times \Sigma$ of 11-dimensional space is as follows,

$$\begin{aligned}
&i_1 = 0,1,2 && \Gamma^{i_1} = \gamma^{i_1} \otimes I_2 \otimes I_2 \otimes \sigma^1 \otimes \sigma^3 \\
&i_2 = 3,4,5 && \Gamma^{i_2} = I_2 \otimes \gamma^{i_2} \otimes I_2 \otimes \sigma^2 \otimes \sigma^3 \\
&i_3 = 6,7,8 && \Gamma^{i_3} = I_2 \otimes I_2 \otimes \gamma^{i_3} \otimes \sigma^3 \otimes \sigma^3 \\
&a = 9,\natural && \Gamma^{a} = I_2 \otimes I_2 \otimes I_2 \otimes I_2 \otimes \sigma^a
\end{aligned} \tag{A.2}$$

where we have introduced the following γ-matrices associated with each of the 3-dimensional symmetric spaces AdS_3, S_2^3, and S_3^3,

$$\begin{aligned}
-i\gamma^0 = \gamma^3 = \gamma^6 = \sigma^\natural &= \sigma^2 \\
\gamma^1 = \gamma^4 = \gamma^7 = \sigma^9 &= \sigma^1 \\
\gamma^2 = \gamma^5 = \gamma^8 &= \sigma^3
\end{aligned} \tag{A.3}$$

Using the defining property of complex conjugation,

$$B\Gamma^M B^{-1} = (\Gamma^M)^* \tag{A.4}$$

the complex conjugation matrix B is given by

$$B = 1 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^1 \tag{A.5}$$

The Majorana condition on the spinors $\zeta^* = B\zeta$ can be solved in terms of the eigenspinors of σ^1 and σ^2 respectively.

B. Geometry of Killing spinors in odd dimensions

In this appendix, we review the geometry of Killing spinors on spheres and Minkowski signature hyperbolic spaces of odd dimensions $d = 2n + 1$.

B.1 Killing spinors on S^d

The 2^n-dimensional Clifford algebra generators of SO(d) γ^i obey $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$, where $i, j = 1, 2, \ldots, d$. The 2^{n+1}-dimensional SO$(d+1)$ Clifford algebra generators Γ^I and the associated chirality matrix $\bar{\Gamma}$ may be constructed out of the generators γ^i by

$$\Gamma^i = \gamma^i \otimes \sigma^1 \qquad \Gamma^{d+1} = I \otimes \sigma^2 \qquad \bar{\Gamma} = I \otimes \sigma^3 \tag{B.1}$$

The sphere may be represented as the coset space $S^d = \mathrm{SO}(d+1)/\mathrm{SO}(d)$, which is maximally symmetric. The Maurer-Cartan 1-form $\omega^{(t)}$ of SO$(d+1)$ provides a flat connection on SO$(d+1)$ with torsion, and obeys,

$$d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0 \qquad \omega^{(t)} = U^\dagger dU \tag{B.2}$$

where U parametrizes SO$(d+1)$ in the spinor representation. Under the subgroup SO(d), the Maurer-Cartan form $\omega^{(t)}_{IJ}$ (with $I, J = 1, 2, \ldots, d, d+1$) decomposes into the canonical orthonormal frame 1-form e_i and the associated torsion-free orthonormal connection ω_{ij}, $\omega^{(t)}_{i\,d+1} = e_i$, and $\omega^{(t)}_{ij} = \omega_{ij}$, so that

$$\omega^{(t)} = \frac{1}{4}\omega^{(t)}_{IJ}\Gamma^{IJ} = \frac{1}{2}e_i\Gamma^i\Gamma^{d+1} + \frac{1}{4}\omega_{ij}\Gamma^{ij} \tag{B.3}$$

The spin connection $\omega^{(t)}$ may be consistently restricted to the $\bar{\Gamma}$-chirality $\eta = \pm 1$ eigenspace. We shall denote this restriction by $\omega^{(t)}_\eta$, and express it in the basis of Γ^I adapted to γ^i,

$$\omega^{(t)}_\eta = \frac{i\eta}{2}e_i\gamma^i + \frac{1}{4}\omega_{ij}\gamma^{ij} \tag{B.4}$$

By construction, this connection also satisfies the Maurer-Cartan equation. The equation for SO$(d+1)$-covariantly constant spinors ε_η on S^d then becomes identical to the Killing spinor equation on S^d, and is given by,

$$\left(d + \omega^{(t)}_\eta\right)\varepsilon_\eta = \left(d + \frac{1}{4}\omega_{ij}\gamma^{ij} + i\frac{\eta}{2}e_i\gamma^i\right)\varepsilon_\eta = 0 \tag{B.5}$$

Its solution space is of maximal rank, and given by

$$\varepsilon_\eta = \frac{1}{2}(I + \eta\bar{\Gamma})U\varepsilon_0 \tag{B.6}$$

where ε_0 is an arbitrary constant Dirac spinor, satisfying $d\varepsilon_0 = 0$.

Under complex conjugation, the $d+1$-dimensional Euclidean signature Dirac matrices behave as follows, (recall that we set $d = 2n+1$),

$$\begin{aligned} (\gamma^i)^* &= (-)^n B_E \gamma^i B_E^{-1} \\ B_E B_E^* &= -(-)^m I \qquad\qquad m = \left[\frac{n}{2}\right] \end{aligned} \tag{B.7}$$

where [ ] denotes the integer part of the argument. Under complex conjugation, and the use of the complex conjugation matrix B_E, the Killing spinor equation becomes,

$$\left(d + \frac{1}{4}\omega_{ij}\gamma^{ij} - i(-)^n \frac{\eta}{2} e_i \gamma^i \right) B_E^{-1}\varepsilon_\eta^* = 0 \tag{B.8}$$

Therefore, the combination $B_E^{-1}\varepsilon_\eta^*$ is again a Killing spinor, but for $\eta \to \eta' \equiv -(-)^n\eta$, and we may identify those two spinors as follows,

$$B_E^{-1}\varepsilon_\eta^* = \tilde{\varepsilon}_{\eta'} \tag{B.9}$$

Whenever n is odd, such as in the case of the spheres S^3 in the present paper, we have $\eta' = \eta$.

B.2 Killing spinors on Minkowski signature AdS_d

The 2^n-dimensional Clifford algebra generators of SO$(1,2n)$ γ^μ obey $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, where $\mu, \nu = 0, 1, \ldots, 2n$. The 2^{n+1}-dimensional SO$(2,2n)$ Clifford algebra generators $\Gamma^{\bar{\mu}}$ and the associated chirality matrix $\bar{\Gamma}$ may be constructed out of the generators γ^μ by

$$\Gamma^\mu = \gamma^\mu \otimes \sigma^1 \qquad\qquad \Gamma^{2n+1} = -iI \otimes \sigma^2 \qquad\qquad \bar{\Gamma} = I \otimes \sigma^3 \tag{B.10}$$

Minkowski signature anti-de Sitter space may be represented as the coset space $AdS_d =$ SO$(2,2n)/$SO$(1,2n)$, which is maximally symmetric. The Maurer-Cartan 1-form $\omega^{(t)}$ of SO$(2,2n)$ provides a flat connection on SO$(2,2n)$ with torsion, and obeys,

$$d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0 \qquad\qquad \omega^{(t)} = U^\dagger dU \tag{B.11}$$

where U parametrizes SO$(2,2n)$ in the spinor representation. Under the subgroup SO$(1,2n)$, the Maurer-Cartan form $\omega^{(t)}_{\bar{\mu}\bar{\nu}}$ (with $\bar{\mu}, \bar{\nu} = 0, 1, 2, \ldots, 2n, 2n+1$) decomposes into the canonical orthonormal frame 1-form e_i and the associated torsion-free orthonormal connection $\omega_{\mu\nu}$, $\omega^{(t)}_{\mu\, d} = e_\mu$, and $\omega^{(t)}_{\mu\nu} = \omega_{\mu\nu}$, so that

$$\omega^{(t)} = \frac{1}{4}\omega^{(t)}_{\bar{\mu}\bar{\nu}}\Gamma^{\bar{\mu}\bar{\nu}} = \frac{1}{2}e_\mu \Gamma^\mu \Gamma^d + \frac{1}{4}\omega_{\mu\nu}\Gamma^{\mu\nu} \tag{B.12}$$

The spin connection $\omega^{(t)}$ may be consistently restricted to the $\bar{\Gamma}$-chirality $\eta = \pm 1$ eigenspace. We shall denote this restriction by $\omega^{(t)}_\eta$, and express it in the basis of $\Gamma^{\bar{\mu}}$ adapted to γ^μ,

$$\omega^{(t)}_\eta = \frac{\eta}{2}e_\mu\gamma^\mu + \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu} \tag{B.13}$$

By construction, this connection also satisfies the Maurer-Cartan equation. The equation for SO$(2,2n)$-covariantly constant spinors ε_η on AdS_d then becomes identical to the Killing spinor equation on AdS_d, and is given by,

$$\Big(d+\omega_\eta^{(t)}\Big)\varepsilon_\eta = \Big(d+\frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu}+\frac{\eta}{2}e_\mu\gamma^\mu\Big)\varepsilon_\eta = 0 \tag{B.14}$$

Its solution space is of maximal rank, and given by

$$\varepsilon_\eta = \frac{1}{2}(I+\eta\bar\Gamma)U\varepsilon_0 \tag{B.15}$$

where ε_0 is an arbitrary constant Dirac spinor, satisfying $d\varepsilon_0 = 0$.

Under complex conjugation, the $d+1$-dimensional Minkowski signature Dirac matrices behave as follows,

$$\begin{aligned} (\gamma^\mu)^* &= -(-)^n B_M \gamma^\mu B_M^{-1} \\ B_M B_M^* &= (-)^m I \qquad\qquad m = \Big[\frac{n}{2}\Big] \end{aligned} \tag{B.16}$$

where [ ] denotes the integer part of the argument. Under complex conjugation, and the use of the complex conjugation matrix B_M, the Killing spinor equation becomes,

$$\Big(d+\frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu}-(-)^n\frac{\eta}{2}e_\mu\gamma^\mu\Big)B_M^{-1}\varepsilon_\eta^* = 0 \tag{B.17}$$

Therefore, the combination $B_M^{-1}\varepsilon_\eta^*$ is again a Killing spinor, but for $\eta\to\eta' = -(-)^n\eta$, and we may identify those two spinors as follows,

$$B_M^{-1}\varepsilon_\eta^* = \tilde\varepsilon_{\eta'} \tag{B.18}$$

Whenever n is odd, as is the case of the AdS_3 in the present paper, we have $\eta' = \eta$.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[5] E. D'Hoker and D.Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, hep-th/0201253.

[6] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and* 1/2 *BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174].

[7] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [hep-th/0403110].

JHEP08(2008)028

[8] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS type IIB interface solutions I: local solution and supersymmetric Janus*, *JHEP* **06** (2007) 021 [arXiv:0705.0022].

[9] E. D'Hoker, J. Estes and M. Gutperle, *Exact half-BPS type IIB interface solutions. II: flux solutions and multi-Janus*, *JHEP* **06** (2007) 022 [arXiv:0705.0024].

[10] D. Bak, M. Gutperle and S. Hirano, *A dilatonic deformation of AdS_5 and its field theory dual*, *JHEP* **05** (2003) 072 [hep-th/0304129].

[11] A. Clark and A. Karch, *Super Janus*, *JHEP* **10** (2005) 094 [hep-th/0506265].

[12] E. D'Hoker, J. Estes and M. Gutperle, *Ten-dimensional supersymmetric Janus solutions*, *Nucl. Phys.* **B 757** (2006) 79 [hep-th/0603012].

[13] O. DeWolfe, D.Z. Freedman and H. Ooguri, *Holography and defect conformal field theories*, *Phys. Rev.* **D 66** (2002) 025009 [hep-th/0111135].

[14] A.B. Clark, D.Z. Freedman, A. Karch and M. Schnabl, *The dual of Janus* $((<:) \leftrightarrow (:>))$ *an interface CFT*, *Phys. Rev.* **D 71** (2005) 066003 [hep-th/0407073].

[15] E. D'Hoker, J. Estes and M. Gutperle, *Interface Yang-Mills, supersymmetry and Janus*, *Nucl. Phys.* **B 753** (2006) 16 [hep-th/0603013].

[16] J. Gomis and C. Romelsberger, *Bubbling defect CFT's*, *JHEP* **08** (2006) 050 [hep-th/0604155].

[17] D. Gaiotto and E. Witten, *Janus configurations, Chern-Simons couplings and the Theta-angle in $N = 4$ super Yang-Mills theory*, arXiv:0804.2907.

[18] D. Gaiotto and E. Witten, *Supersymmetric boundary conditions in $N = 4$ super Yang-Mills theory*, arXiv:0804.2902.

[19] E. D'Hoker, J. Estes and M. Gutperle, *Gravity duals of half-BPS Wilson loops*, *JHEP* **06** (2007) 063 [arXiv:0705.1004].

[20] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev.* **D 60** (1999) 125006 [hep-th/9904191].

[21] S. Yamaguchi, *Bubbling geometries for half BPS Wilson lines*, *Int. J. Mod. Phys.* **A 22** (2007) 1353 [hep-th/0601089].

[22] J. Gomis and F. Passerini, *Holographic Wilson loops*, *JHEP* **08** (2006) 074 [hep-th/0604007].

[23] O. Lunin, *On gravitational description of Wilson lines*, *JHEP* **06** (2006) 026 [hep-th/0604133].

[24] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple $M2$-branes*, *Phys. Rev.* **D 77** (2008) 065008 [arXiv:0711.0955].

[25] J. Bagger and N. Lambert, *Comments on multiple $M2$-branes*, *JHEP* **02** (2008) 105 [arXiv:0712.3738].

[26] A. Gustavsson, *Selfdual strings and loop space Nahm equations*, *JHEP* **04** (2008) 083 [arXiv:0802.3456].

[27] H.J. Boonstra, B. Peeters and K. Skenderis, *Brane intersections, anti-de Sitter spacetimes and dual superconformal theories*, *Nucl. Phys.* **B 533** (1998) 127 [hep-th/9803231].

[28] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Supersymmetry of rotating branes*, *Phys. Rev.* **D 59** (1999) 025001 [hep-th/9809065].

JHEP08(2008)028

[29] J. de Boer, A. Pasquinucci and K. Skenderis, *AdS/CFT dualities involving large* $2D$ $N=4$ *superconformal symmetry*, *Adv. Theor. Math. Phys.* **3** (1999) 577 [hep-th/9904073].

[30] J.P. Gauntlett, N. Kim and D. Waldram, *Supersymmetric* AdS_3, AdS_2 *and bubble solutions*, *JHEP* **04** (2007) 005 [hep-th/0612253].

[31] J.P. Gauntlett, O.A.P. Mac Conamhna, T. Mateos and D. Waldram, *New supersymmetric* AdS_3 *solutions*, *Phys. Rev.* **D 74** (2006) 106007 [hep-th/0608055].

[32] J.P. Gauntlett and N. Kim, *Geometries with Killing spinors and supersymmetric AdS solutions*, arXiv:0710.2590.

[33] O. Lunin, 1/2-*BPS states in M-theory and defects in the dual CFTs*, *JHEP* **10** (2007) 014 [arXiv:0704.3442].

[34] E. D'Hoker, J. Estes, M. Gutperle, D. Krym and P. Sorba, *Half-BPS supergravity solutions and supergroups*, in preparation.

[35] E. D'Hoker and R. Jackiw, *Liouville field theory*, *Phys. Rev.* **D 26** (1982) 3517; *Space translation breaking and compactification in the Liouville theory*, *Phys. Rev. Lett.* **50** (1983) 1719;
E. D'Hoker, D.Z. Freedman and R. Jackiw, SO(2, 1) *invariant quantization of the Liouville theory*, *Phys. Rev.* **D 28** (1983) 2583.

[36] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in* 11 *dimensions*, *Phys. Lett.* **B 76** (1978) 409.

JHEP08(2008)028